THE

ELECTRIC CIRCUIT

BY

V. KARAPETOFF

SECOND EDITION

REWRITTEN, ENLARGED AND ENTIRELY RESET

SEVENTH IMPRESSION

McGRAW-HILL BOOK COMPANY, INC.

NEW YORK AND LONDON

1912

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE TO THE FIRST EDITION

This pamphlet, together with the companion pamphlet entitled *The Magnetic Circuit*, is intended to give a student in electrical engineering the theoretical elements necessary for calculation of the performance of dynamo-electric machinery and of transmission lines. The advanced student must be taught to treat every electric machine as a particular combination of electric and magnetic circuits, and to base its performance upon the fundamental theoretical relations rather than upon a separate "theory" established for each kind of machinery, as is often done.

The first chapter is devoted to a review of the direct-current circuit, the next four chapters treat of sine-wave alternating-current circuits, and the last two chapters give the fundamental properties of the electrostatic circuit. All the important results and methods are illustrated by numerical problems of which there are over one hundred in the text. The pamphlet is *not* intended for a beginner, but for a student who has had an elementary descriptive course in electrical engineering and some simple laboratory experiments.

The treatment is made as far as possible uniform, so that the student sees analogous relations in the direct-current circuit, in the alternating-current circuit, in the electrostatic circuit and finally in the magnetic circuit. All matter of purely historical or academic interest, not bearing directly upon the theory of electric machinery, has been left out. An ambitious student will find a more exhaustive treatment in the works mentioned at the end of the pamphlet.

The electrostatic circuit is treated in accordance with the modern conception of elastic displacement of electricity in dielectrics. No use has been made of the action of electric charges at a distance, or of the electrostatic system of units. The volt-ampere-ohm system of units is used for electrostatic calculations,

in accordance with Professor Giorgi's ideas (see a paper by Professor Ascoli in Vol. I of the *Transactions of the International Electrical Congress*, St. Louis, 1904). Those familiar with Oliver Heaviside's writings will notice his influence upon the author, particularly in Arts. 22 and 23,* where an attempt is made at a rational electrostatic nomenclature.

Many thanks are due to the author's friend and colleague, Mr. John F. H. Douglas, instructor in electrical engineering in Sibley College, who read the manuscript and the proofs, checked the answers to the problems and made many excellent suggestions for the text.

CORNELL UNIVERSITY, ITHACA, N. Y.
August, 1910.

PREFACE TO THE SECOND EDITION

THE first edition of this book was issued as a pamphlet of some 85 pages which the author used for two years in his classes to supplement some other texts. In its present edition, the book is made independent of these texts, so that its size had to be more than doubled. The book has been practically rewritten, and completely reset in type. All the cuts are new. The topics are treated somewhat more in detail, and a large number of practical problems are provided. The new topics added are: the resistance of conductors of variable cross-section, the electrical relations in polyphase systems, performance characteristics of the transmission line, transformer and induction motor and the permittance (electrostatic capacity) of transmission lines.

In the treatment of alternating currents by means of complex quantities, particular attention is paid to the trigonometric form $E(\cos\theta + j\sin\theta)$ of the expression for a vector. In fact, the transmission line, the transformer, and the induction motor to some extent, are treated in this trigonometric form. The author trusts that the reader will find this somewhat novel treatment more convenient in numerical applications than the usual form $e + je'$.

* Chapter 14 in the second edition.

Since the appearance of the first edition, the author has been encouraged by some of his colleagues in his treatment of the electrostatic circuit in the ampere-ohm system of units, a treatment which involves the use of permittances in farads and elastances in darafs. He has extended this treatment to the calculation of capacity of cables and transmission lines. The students grasp this mode of presentation much more readily than the old-fashioned way, based upon the law of inverse squares and electric charges acting at a distance. The purpose of the present treatment is to impress them with the idea of a continuous action in the medium itself and with the rôle of the dielectric.

Mr. F. R. Keller of the electrical department of Columbia University has read and corrected the manuscript and the proofs of the second edition, and checked the answers to the new problems. The author wishes to express sincere appreciation of his painstaking, faithful and competent work. The author is also indebted to Mr. John F. H. Douglas for critically reading the galley proof of the second edition.

CORNELL UNIVERSITY, ITHACA, N. Y.
May, 1912.

CONTENTS

SUGGESTIONS TO TEACHERS

(1) This book is intended to be used as a text in a course which comprises lectures, recitations, computing periods and home work. Purely descriptive matter has been omitted or only suggested, in order to allow the teacher more freedom in his lectures and to permit him to establish his own point of view. Some parts of the book are more suitable for recitations, others for reference in the designing room, others again as a basis for discussion in the lecture room, or for brief theses.

(2) Different parts of the book are made as much as possible independent of one another, so that the teacher can schedule them as it suits him best. Moreover, most chapters are written according to the concentric method, so that it is not necessary to finish one chapter before starting on the next. One can cover the subject in an abridged manner, omitting the last parts of some chapters.

(3) The problems given at the end of nearly every article are an integral part of the book, and should under no circumstances be omitted. There is no royal way of obtaining a clear understanding of the underlying physical principles, and of acquiring an assurance in their practical application, except by the solution of numerical examples.

(4) The book contains comparatively few sketches, in order to give the student an opportunity to illustrate the important relations by sketches of his own. Making sketches, diagrams and drawings of electric circuits and machines to scale should be one of the important features of the course, even though it may not be popular with some analytically inclined students. Mechanical drawing develops precision of judgment, and gives the student a knowledge that is tangible and concrete.

(5) The author has avoided giving definite numerical data, coefficients and standards, except in problems, where they are indispensable and where no general significance is ascribed to such

data. His reasons are: (*a*) Numerical coefficients obscure the general exposition. (*b*) Sufficient numerical coefficients and design data will be found in good electrical hand-books and pocket-books, one of which ought to be used in conjunction with this text. (*c*) The student is likely to ascribe too much authority to a numerical value given in a text-book, while in reality many coefficients vary within wide limits according to the conditions of a practical problem, and with the progress of the art. (*d*) Most numerical coefficients are obtained in practice by assuming that the phenomenon in question occurs according to a definite law, and by substituting the available experimental data into the corresponding formula. This point of view is emphasized throughout the book, and gives the student the comforting feeling that he will be able to obtain the necessary numerical constants when confronted by a definite practical situation.

(6) The treatment of the electrostatic circuit is made as much as possible analogous to that of the electrodynamic circuit. The teacher will find it advisable to make his students perfectly familiar with the use of Ohm's law for ordinary electric circuits before starting on the electrostatic circuit. The student should solve several numerical examples involving voltages and voltage gradients, currents and current densities, resistances, resistivities, conductances and conductivities. He will then find very little difficulty in mastering the electrostatic circuit, and from these two the transition to the magnetic circuit, treated in the companion book, is very simple indeed. The following table shows the analogous quantities in the three kinds of circuits.

Electrodynamic	*Electrostatic*	*Magnetic*
Voltage or e.m.f.	Voltage or e.m.f.	Magnetomotive force
Voltage gradient (or electric intensity)	Voltage gradient (or electric intensity)	M.m.f. gradient (or magnetic intensity)
Electric current	Dielectric flux	Magnetic flux
Current density	Dielectric flux density	Magnetic flux density
Resistor	Elastor	Reluctor
Resistance	Elastance	Reluctance
Resistivity	Elastivity	Reluctivity
Conductor	Permittor (condenser)	Permeator
Conductance	Permittance (capacity)	Permeance
Conductivity	Permittivity (dielectric constant)	Permeability

LIST OF PRINCIPAL SYMBOLS.

The following list comprises most of the symbols used in the text. Those not occurring here are explained where they appear. When, also, a symbol has a use different from that stated below, the correct meaning is given where the symbol occurs.

Symbol.	Meaning.	Page where defined or first used.
a	Radius of conductor of transmission line	176
a	Radius of core of cable	171
A	Cross-section	13
b	Interaxial distance between conductors of transmission line	176
b	Radius of inner surface of cable sheathing	171
b	Susceptance	75
C	Constant	54, 72
C	Permittance or electrostatic capacity	147
D	Dielectric flux density	154
D_{max}	Rupturing flux density	165
e	Electromotive force	1
e	Instantaneous value of voltage	34
e	Horizontal component of vector of e.m.f.	83
e'	Vertical component of vector of e.m.f.	83
e_l	Local source of e.m.f.	3
e_t	Terminal voltage	3
E	Effective value of alternating voltage	48
$\dot{E}$	Vector of the voltage E	83
E_m	Maximum value of voltage	34
f	Frequency of alternating current or voltage	33
F	Mechanical force	60
g	Conductance	2
g_{eq}	Equivalent conductance	8
G	Voltage gradient or electric intensity	16
G_{max}	Rupturing voltage gradient	165
h	Elevation of conductor above the ground	181
h	Head of fluid	10
h	Instantaneous value of harmonic	54
i	Current	1
i	Horizontal component of vector of current	87
i	Instantaneous value of current	33
i'	Vertical component of vector of current	87
I	Effective value of alternating current	48
$\dot{I}$	Vector of the current I	88

Symbol.	Meaning.	Page where defined or first used.
α	Angle	43, 94
α	A ratio	189
α	Temperature coefficient	5
γ	Conductivity	14
ϵ	Base of natural system of logarithms	72
θ	Difference of temperature	2
θ	Phase angle	82
θ_1	Angle of incidence of current	28
θ_1	Angle of incidence of dielectric flux	163
θ_2	Angle of refraction of current	28
θ_2	Angle of refraction of dielectric flux	163
κ	Permittivity	151
κ_a	Permittivity of air	151
ρ	Resistivity	13
σ	Circle coefficient or dispersion factor of induction motor	138
σ	Elastivity	152
σ_a	Elastivity of air	152
τ	Time constant of a circuit	72
ϕ	Phase angle	34
Φ	Magnetic flux	114
Φ_m	Maximum value of magnetic flux	114
ψ	Angle	121
ω	Angle	52
Ω	Angle	52

THE ELECTRIC CIRCUIT

CHAPTER I

FUNDAMENTAL ELECTRICAL RELATIONS IN DIRECT-CURRENT CIRCUITS

1. The Volt, the Ampere, the Ohm, and the Mho. The student is supposed to be familiar with Ohm's law, both theoretically and from his laboratory experience. A brief synopsis of the law, given below, is intended to refresh the relations in his mind, and to establish a point of view which permits of extending these relations to alternating-current circuits. Moreover, the law is presented in a form applicable to magnetic and dielectric circuits.

When the current in a conductor is steady and there are no local electromotive forces within the conductor, the value of the current is proportional to the voltage between the terminals of the conductor. This is an experimental fact, called Ohm's law. The word "conductor" is used here in the sense of "the part of the circuit under consideration." It may consist of two or more distinct physical conductors. Considering the electromotive force e as the cause of the current i, this law merely states that *the effect is proportional to the cause*, or

$$e = r \cdot i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the coefficient of proportionality r is called the resistance of the conductor. When the current is expressed in amperes, and the electromotive force in volts, the resistance r is measured in units called ohms.

Ohm's law is sometimes written in the form

$$i = g \cdot e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the coefficient of proportionality

$$g = 1/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is called the *conductance* of the conductor. The reason for this name is easy to see: The resistance r shows how *difficult* it is to force a unit current through a given conductor, while its reciprocal g shows how *easy* it is to produce the same current in the same conductor. Conductances are measured in units called *mhos*, one mho being the reciprocal of one ohm. Hence, a resistance of one ohm represents at the same time a conductance of one mho; a resistance of two ohms has a conductance of one-half mho, etc. Increasing the resistance of a winding from 4 to 5 ohms reduces its conductance from 0.25 to 0.20 mho.

It will be seen below that in some problems it is convenient to use conductances instead of resistances. Both are fundamental, and there is no reason why Ohm's law should not have been expressed originally by eq. (2) instead of (1).

With our present meager knowledge of the true nature of electrical phenomena, it is well-nigh impossible to give a clear physical meaning of the quantities under discussion without resorting to analogies. For instance, the flow of current through a conductor may be compared to the flow of heat through a rod; the voltage or difference of electric potential is analogous to the difference of temperature θ at the ends of the rod, and the electric current to the quantity of heat Q passing through a cross-section of the rod in unit time (the rate of flow of heat). The ratio of θ to Q is sometimes called the thermal resistance of the rod.[1]

Again, the phenomenon of the flow of electricity is somewhat analogous to the flow of water through pipes. The hydraulic head may be likened to the voltage, and the rate of discharge of water to the current. With very low velocities, in capillary tubes, the discharge is proportional to the head, so that eqs. (1) and (2) hold true for the flow of water.

Whatever the reasons which have led originally to the choice of the magnitudes of the ampere, the ohm, and the volt, these units may be considered at present, for all practical and most theoretical purposes, as arbitrary units, like the foot, the pound, or

[1] It even has been proposed to measure this resistance in thermal ohms or *thohms*.

the meter. Their values have been established by an international agreement, whence the name, international electrical units. These units are represented by concrete standards with minutely specified dimensions and properties; the ohm by a column of mercury, the ampere by a silver voltameter, and the volt by a standard cell. It is understood, of course, that only two out of the three units need to be standardized, the third being determined either as their product, or their ratio. It has been decided by international agreement to consider the ampere and the ohm as fundamental units, the volt being derived from them. Hence the present system of practical electrical units is properly called *the ampere-ohm system*. This fact does not preclude, of course, the use of standard cells as secondary standards.

The ampere, the volt, and the ohm are connected by simple multipliers (powers of 10) with the absolute electromagnetic units (the C.G.S. system of units). It is conceded at present by some prominent physicists that the choice of the units was not quite fortunate, according to our present understanding of the electromagnetic relations. Since, however, it is too late to change these units, it is better to consider them as arbitrary, and not connected in any way with the magnitudes of the centimeter, the gram, and the second.

In applying Ohm's law to practical problems, it must be clearly remembered that e represents the *net* voltage acting between the ends of the conductor r. This is important when the circuit contains sources of counter-electromotive force, such as electric batteries, or motors. Let, for instance, the total resistance of a circuit, connected across the terminals of a generator, be 12 ohms, and let the terminal voltage of the generator be 120 volts. Then the current is equal to 10 amperes, provided that there are no counter-electromotive forces in the circuit. Let, however, the circuit contain a storage battery of, say, 24 volts, connected so as to be charging, that is, opposing the applied voltage. The current in the circuit is now only $(120 - 24)/12 = 8$ amp., the value $120 - 24 = 96$ being the *net* voltage in the external circuit. Should the terminals of the battery be reversed, so as to help the generator voltage, the current would increase to $(120 + 24)/12 = 12$ amp.

Thus, when there is an external or *local* source of electromotive force, say e_l, within a conductor, the *terminal* voltage e_t

between the ends of the conductor is added algebraically to e_t, so that we have, instead of eq. (1),

$$e_t + e_l = i \cdot r, \qquad (4)$$

where e_l is considered positive when in the same direction as e_t. In the foregoing numerical example the counter-e.m.f. is therefore considered negative.

In numerical computations it is sometimes convenient to use multiples and submultiples of the units originally agreed upon, in order to avoid large numbers or very small fractions. This is accomplished by adding to the names of the original units certain Greek prefixes for the multiples, and Latin prefixes for the submultiples. These prefixes are as follows:

deca....ten	deci......one tenth
hecto...one hundred	centi......one hundredth
kilo.....one thousand	milli......one thousandth
mega...one million	micro.....one millionth.

For instance, instead of 10,000 amperes one may say or write 10 kiloamperes; instead of 0.0003 volt one may say 0.3 millivolt, or 300 microvolts, etc. Another way to avoid very large or very small numbers is to use 10 to the proper power as a multiplier. For instance, one may speak of a resistance of 7×10^{-6} ohm, of a conductance equal to 5×10^7 mhos, etc.

Prob. 1. In order to determine the resistance of the armature of an electrical machine, a direct current is sent through it and the drop of voltage is measured between the brushes. The following are the readings:

Volts................	0.44	0.73	1.00	1.33	1.73
Amperes.............	8.1	12.9	18.1	24.0	31.0

What is the most probable value of the resistance? Hint: Take an average of the ratios, or better, plot the volts against the amperes as abscissæ and draw a straight line through the origin.

Ans. 0.0559 ohm.

Prob. 2. The resistance of a transmission line is 1.2 ohms. What voltage is necessary at the generating end in order to produce a current of 75 amp. (a) when the line is short-circuited at the receiving end; (b) when a pressure of 500 volts must be maintained at the receiving end?

Ans. 90 volts; 590 volts.

Prob. 3. The armature resistance of a 250-volt generator is 0.025 ohm. At what current will the voltage drop in the armature be equal to 4 per cent of the terminal voltage?

Ans. 400 amp.

Prob. 4. The conductance of a bath of molten metal is 5 kilomhos; what voltage is required to send a current of 7×10^4 amp. through it?

Ans. 14 volts.

Prob. 5. The coil of a regulating electromagnet of 50 ohms resistance is connected across a 110-volt line; the voltage of the line fluctuates by $\pm$ 10 per cent. In order to make the regulating mechanism more sensitive, that is, in order to accentuate the fluctuations of current in it, a counter-e.m.f. storage battery of negligible resistance is connected in series with the coil. What must be the voltage of the battery if the current in the coil at 120 volts must be twice that at 100 volts?

Ans. 80 volts.

2. Temperature Coefficient. The resistance of all metals and of practically all alloys increases with the temperature, according to a rather complicated law. Within the usual limits of temperature the increase in resistance is nearly proportional to the temperature rise; in other words, the relation between the resistance of a conductor and its temperature is represented by a straight

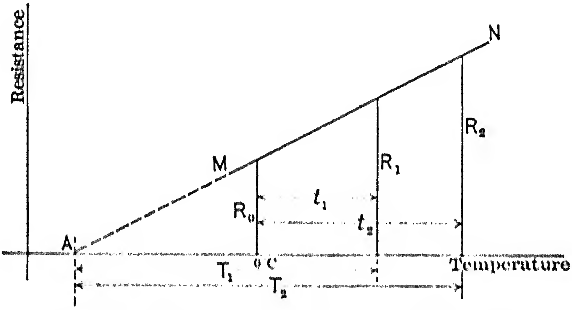

FIG. 1. The relation between the resistance and the temperature of metals.

line MN (Fig. 1). Let the resistance at 0° C. be R_0 ohms; then the resistance at some temperature t° C. is

$$R_t = R_0(1 + \alpha t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where α is called the *temperature coefficient* of the material. For the values of α for various materials see an electrical handbook. For the most important material, copper, $\alpha = 0.0042$; in other words, the resistance of a copper conductor increases by 0.42 per cent for each degree centigrade, considering the resistance at 0° C. as 100 per cent.

The formula given below is sometimes more convenient in computations than formula (5). Assume that the same straight-

line law (Fig. 1) holds for low temperatures, and let temperatures be measured from the point A at which the straight line crosses the axis of abscissæ. Denoting temperatures from this point by T, we have, for any two temperatures,

$$R_1/R_2 = T_1/T_2. \qquad (6)$$

With this formula it is not necessary to refer computations to the resistance at 0° C. The point A is found from the condition $R_t = 0$, from which, according to eq. (5), $t_A = -1/\alpha$. Thus, for any temperature,

$$T = t + 1/\alpha. \qquad (7)$$

For copper, $T = t + 238.1$; that is, point A lies 238.1° C. below the freezing point of water. This does not mean that the resistance of copper actually varies according to this law at such temperatures; A is merely a fictitious point through the position of which it is convenient to express the equation of the full-drawn part of the straight line in Fig. 1.

Let, for instance, the resistance of the winding of an electric machine be 0.437 ohm at the room temperature of 22° C. After the machine has been run for several hours the resistance of the same winding is found to be 0.482 ohm, with the room temperature unchanged. Let it be required to calculate the final temperature of the winding from the increase in its resistance. We have $T_1 = 238.1 + 22 = 260.1$, and according to eq. (6) the unknown final temperature $T_2 = 260.1 \times (482/437) = 286.9$, or $t_2 = 286.9 - 238.1 = 48.8°$ C. Two other practical formulæ for temperature rise will be found in Appendix E of the *Standardization Rules* of the American Institute of Electrical Engineers. These rules are reprinted in most electrical handbooks and pocketbooks. See also a convenient method given in problem 2 below.

Prob. 1. The resistance of a conductor increases by 31 per cent from 23° to 75° C. What is α in formula (5)? Ans. 0.00691.

Prob. 2. The relation between the resistance and the temperature of copper conductors is easily obtained on an ordinary slide rule, as follows: On the lower movable scale mark 0° C. on division 238; 10° C. on division 248, and so on. Set a known resistance on the lower fixed scale, and bring the corresponding temperature opposite. Then the resistance at any other temperature is read opposite the corresponding division on the temperature scale. Give an explanation of this method.

Prob. 3. Prove the formulæ for R_{i+r} and r in the above-mentioned Standardization Rules.

3. Resistances and Conductances in Series and in Parallel. When resistances are connected in series, the total resistance of the circuit is increased. This can be more easily seen by resorting to analogies. For instance, if the length of a pipe carrying a fluid be increased, the frictional resistance to the flow becomes greater; in like manner, a long rod offers a more difficult path for the passage of heat than a short one In the electric circuit, the equivalent resistance of two conductors in series is equal to the sum of their individual resistances, as is shown below. This follows from the experimental fact that electricity in its flow behaves like an incompressible fluid; that is, the same quantity of it must pass in a given time interval through all the cross-sections of a circuit.

Let two conductors, r_1 and r_2, be connected in series across a source of voltage e, and let a current i flow through them. Part of the total voltage e is spent in overcoming the resistance of the first conductor, the rest in overcoming that of the second conductor. But, according to Ohm's law, when the conditions are steady, the voltage across the first conductor, $e_1 = i \cdot r_1$; the voltage across the second is $e_2 = i \cdot r_2$. Adding these two equations gives the total voltage

$$e = e_1 + e_2 = i(r_1 + r_2).$$

An equivalent resistance, r_{eq}, by definition, is one which, with the same total voltage e, allows the same current i to pass through the circuit, as the combination of the given conductors. Hence,

$$e = i \cdot r_{eq}.$$

Comparing the two foregoing equations gives

$$r_{eq} = r_1 + r_2. \qquad (8)$$

The law is true for any number of conductors in series; it may be proved by successively combining them into groups of two.

When several conductors are connected in parallel, the voltage across them is common to all the branches, so that we have

$$\left.\begin{array}{l} e = i_1 \cdot r_1 \\ e = i_2 \cdot r_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}, \qquad (9)$$

where i_1, i_2, . . . are the currents in the separate branches. The total current is equal to the sum of the currents in the separate

branches, because electricity behaves in its flow like an incompressible fluid. Thus, the equivalent resistance, r_{eq}, is determined by the condition

$$e = (i_1 + i_2 + \text{etc.}) \cdot r_{eq}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Substituting the values of i_1, i_2, etc., from (9) into (10) and canceling e, gives

$$1/r_{eq} = 1/r_1 + 1/r_2 + \text{etc.}, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

or, in words: when two or more conductors are connected in parallel, the reciprocal of the equivalent resistance is equal to the sum of the reciprocals of the individual resistances.

We have defined conductance as the reciprocal of resistance, so that eq. (11) may be written also in the form

$$g_{eq} = g_1 + g_2 + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It will thus be seen that it is convenient to use conductances in parallel circuits and resistances in series circuits. The simple rule is: *Resistances are added in series; conductances are added in parallel.* This rule follows directly from the physical concept of resistance and conductance.

Prob. 1. Prove that when two conductors, 1 and 2, are in parallel

$$i_1/i_2 = g_1/g_2 = r_2/r_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and that when they are in series

$$e_1/e_2 = r_1/r_2 = g_2/g_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Prob. 2. Show that when two resistances are in parallel the equivalent resistance

$$r_{eq} = r_1r_2/(r_1 + r_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and that for two conductances in series

$$g_{eq} = g_1g_2/(g_1 + g_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Prob. 3. Two resistances, $r_1 = 5$ ohms and $r_2 = 7$ ohms, are connected in series. Resistance r_1 is shunted by a comparatively high resistance $R_1 = 100$ ohms; r_2 is shunted by a resistance $R_2 = 50$ ohms. What is the equivalent resistance of the whole combination? Solution:

Equivalent conductance of r_1 and R_1 is $0.2 + 0.01 = 0.21$ mho;
Equivalent resistance of r_1 and R_1 is $1/0.21 = 4.76$ ohms;
Equivalent conductance of r_2 and R_2 is
$0.1429 + 0.0200 = 0.1629$ mho
Equivalent resistance of r_2 and R_2 is $1/0.1629 = 6.14$ ohms.
Ans. $4.76 + 6.14 = 10.90$ ohms

Prob. 4. Four resistances, $r_1 = 1.2$, $r_2 = 1.7$, $R = 25$, and $r_e =$ 750 ohms, are connected as shown in Fig. 2. The generator voltage

between the points A and B is 500 volts. Determine the current through the resistance R and the voltage across this resistance.[1] Solution: Combine the resistances r_2 and R into one; determine the conductance $1/(R + r_2)$, and combine it with the leakage conductance $1/r_0$. De-

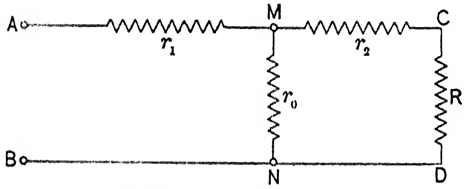

FIG. 2. A series-parallel combination of resistances.

termine the equivalent resistance between the points M and N, and the total resistance between A and B. Having found the total current, subtract from the generator voltage the voltage drop in the part AM of the line. This will give the voltage across MN, and consequently the value of the leakage current. After this, the drop in r_2 is determined, and thus the voltage across the resistance R is found.

Ans. 447.3 volts; 17.8 amps.

Prob. 5. The armature winding of a direct-current machine (Fig. 3) consists of 108 coils; the conductance of each coil is 61 mhos. The coils are connected in series in such a way that the circuit is closed upon itself. Two positive and two negative brushes are placed alternately at four equidistant points of the winding, so as to divide it into four branches in parallel. The two positive brushes are connected together, as are also the two negative brushes. What is the equivalent resistance of the armature between the terminals of the machine?

Ans. 0.1106 ohm.[2]

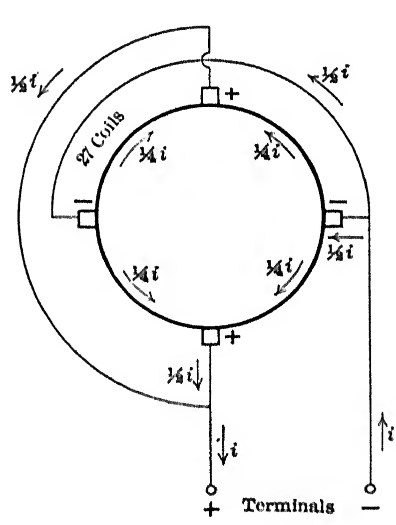

FIG. 3. A four-pole multiple winding.

[1] This combination represents a transmission line the resistance of which is $r_1 + r_2$; the useful load resistance is represented by R, and the leakage resistance by r_0. The problem in a generalized form is of great importance in the theory of alternating-current circuits (see Figs. 41 and 42).

[2] For details of armature windings see the author's *Experimental Electrical Engineering*, Vol. 2, chap. 30.

Prob. 6. Two equal resistances of r ohms each are connected in series. When one of them is shunted by an unknown resistance R, the total resistance of the combination decreases by 10 per cent. Find the value of R. Ans. $4r$.

4. Electric Power. The electric power (energy per unit time) converted into heat in a conductor is found by experiment to be proportional to the resistance of the conductor and to the square of the current (Joule's law). The practical unit of power, the watt, is so selected that the coefficient of proportionality is unity, or the power

$$P = i^2 r = i^2/g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Either i or r may be eliminated from this expression, using Ohm's law. This gives three more expressions for power:

$$P = e \cdot i = e^2/r = e^2 g. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

All these expressions are used in practice, depending upon which quantities are known in a particular case.

The expression $e \cdot i$ is the fundamental one; it is analogous to the expression $h \cdot q$ for the power lost by friction in a pipe in which a fluid is in uniform motion. In the pipe, the energy lost per unit time is equal to the rate of discharge q times the head h lost in friction; in other words, it is equal to the *quantity factor* times the *intensity factor*. In an electric circuit the current i is the quantity factor, while the voltage drop e is the intensity factor.

If P in eqs. (17) and (18) is expressed in kilowatts, or in megawatts, a numerical factor equal to 10^{-3} or 10^{-6} respectively is introduced on the right-hand side of the equation. Sometimes the output of a motor is measured in horse-power; the English horse-power is equal to 746 watts, while the metric horse-power is 736 watts. It is strongly recommended by the International Electrotechnical Commission that the odd and superfluous unit "horse-power" be dropped altogether and that mechanical power be expressed also in watts (or kilowatts). This means that electric motors as well as generators should be rated in kilowatts.

Sometimes the duty of a machine is expressed in kilogrammeters per second; the conversion ratio to watts is:

$$1 \text{ kg.-m. per second} = 9.806 \text{ watts.}$$

In many cases, however, it is not necessary to introduce either kilogram-meters or calories, since mechanical, thermal, and electrical energy can all be expressed in joules (watt-seconds).

If a conductor contains a local e.m.f. e_l, the power communicated to this part of the circuit, between its terminals, is equal to $e_t \cdot i$, where e_t is the terminal voltage. But the power i^2r converted into heat may be either smaller or larger than e_ti, depending upon the polarity or direction of e_l. Multiplying both sides of eq. (4) by i, we find that the power

$$P = e_ti + e_li = i^2r. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Let e_l be positive, that is, in the same direction as e_t (for instance, an extra battery or generator connected into the circuit to boost the voltage); the power i^2r converted into heat is in this case larger than e_ti, because the power supplied by the local source of e.m.f. is also converted into heat. If, however, e_l is negative, that is, if it acts as a counter-e.m.f. (which is usually the case in practice), the power converted into heat is smaller than e_ti. In this case the power e_li is communicated to the local source of e.m.f. If this source is a storage battery, the energy is stored in chemical form, and may be made available at a later time; if it is a motor, the energy is converted into mechanical work on the motor shaft. Let, for instance, the voltage at the terminals of a circuit be 110 volts, and let the counter-e.m.f. of a motor in the circuit be 100 volts; assume the current through the circuit to be 20 amp. Then the voltage drop due to resistance in the conductors is only 10 volts, and the power converted into heat is 200 watts. The power communicated to the motor is 2000 watts, and the total power supplied to the circuit is 2200 watts.

The unit of electrical energy is the watt-second, or joule. When the heat dissipated in a conductor must be expressed in thermal units, use the relation

$$1 \text{ kg.-calorie} = 4186 \text{ joules}.$$

Prob. 1. The armature current of a 220-volt direct-current motor at a certain load is 63 amp., and the armature resistance is 0.14 ohm. How much electric power is converted into mechanical form, and what is the torque developed by the armature if the speed is 1050 r.p.m.?

Ans. 13.3 kw.; 12.3 m.-kg.

Prob. 2. If the currents in the shunted resistances R_1 and R_2 (problem 3, Art. 3) represent pure loss of power, what is the efficiency of the whole arrangement? Solution: Let the voltage across the resistances r_1 and

R_1 be e. Then the voltage across r_2 and R_2 is $e \cdot (6.14/4.76) = 1.29\,e$. Hence, the useful power is $e^2/5 + (1.29\,e)^2/7 = 0.438\,e^2$ watts. The power lost in the resistances R_1 and R_2 is $e^2/100 + (1.29\,e)^2/50 = 0.0433\,e^2$ watts. The efficiency is $43.8/(43.8 + 4.33) = 90$ per cent.

Prob. 3. The heating element of a 110-volt electric kettle must be designed so that it will heat 1.5 liters (1 liter = 1 cu. decimeter) of water at a rate of 10° C. per minute, assuming no losses by radiation. What are the resistance of the element and the rated current of the utensil?

Ans. 11.6 ohms; 9.5 amp.

Prob. 4. It is required to calculate the exciting current i, the number of turns n per pole, and the resistance r per turn of a field coil of a 5000-kw. 6-pole turbo-alternator, from the following data: The excitation required at the rated load is 9000 amp.-turns per pole; at short overloads 12,000 amp.-turns per pole are needed. The external area of the field coil is 280 sq. dm.; in continuous service, 4 sq. cm. of cooling surface must be allowed per watt converted into heat, in order to avoid overheating the coils. The exciter voltage is 125, and during the overload about 10 per cent of this voltage must be absorbed in the field rheostat, as a margin. Hint: Solve the following three equations; $in = 9000$; $i^2rn = 28{,}000/4$; $(12{,}000/n)nr = 0.9 \times 125/6$.

Ans. 500 amp.; 18 turns; 1.562×10^{-3} ohms.

CHAPTER II

FUNDAMENTAL ELECTRICAL RELATIONS IN DIRECT-CURRENT CIRCUITS — (*Continued*)

5. Resistivity and Conductivity. A cylindrical conductor may be considered as a combination of *unit conductors* in series and in parallel. For instance, a wire 12 m. long and having a cross-section of 70 sq. mm. may be regarded as composed of $70 \times 12 = 840$ unit conductors, each of one square millimeter cross-section, and one meter long. These unit conductors are first combined into sets of 70 in parallel, and then the 12 sets are connected in series. The resistance of such a unit conductor, made of copper, and at a temperature of 0° C., is about 0.016 ohm. A set of 70 unit conductors in parallel has $\frac{1}{70}$ of the resistance of one, because the current is offered 70 paths, instead of one; twelve sets connected in series offer twelve times the resistance of one set. Therefore, the resistance of the given conductor is $(0.016/70) \times 12 = 0.002743$ ohm.

Each material is characterized by the resistance of a unit conductor made out of it. The resistance of a unit conductor at a specified temperature is called the *resistivity*[1] of the material and is denoted by ρ. Thus, the resistance of a conductor of a length l and cross-section A is

$$r = \rho \cdot l/A. \qquad (20)$$

The numerical value of ρ depends upon the units of length and resistance used. A unit conductor may, for instance, have a cross-section of one square millimeter, and may be one meter, or one kilometer long; or it may be a centimeter cube. In the English system it may have a cross-section equal to one circular mil, and a length of one foot, one thousand feet, one mile, or any such specified length. Besides, the resistivity may be expressed in ohms, megohms, or microhms. In each case, the unit of resistance and the units in which the dimensions of l and A are expressed in

[1] The older name is "specific resistance."

formula (20) are selected so as to suit the convenience of the user of the formula. For the values of ρ for different materials see one of the various handbooks and pocketbooks for electrical engineers.

In some cases it is more convenient to use the conductance of the unit conductor, instead of its resistance. The conductance of a unit conductor, at the specified temperature, is called the *electric conductivity* (or specific conductance) of the material; the conductivity is the reciprocal of the resistivity of the same material. Denoting this conductivity by γ, we have $\gamma = 1/\rho$. By reasoning similar to that given above, we find that the conductance of a conductor having the dimensions l and A is

$$g = \gamma \cdot A/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Prob. 1. The resistivity of aluminum equals 2.66 microhms per cubic centimeter; what is its conductivity per mil-foot? Solution: The resistance of a conductor one foot long and having a cross-section equal to one circular mil is $2.66 \times 10^{-6} \times 197{,}300 \times 30.48 = 16$ ohms; where $197{,}300 = (1000/2.54)^2 \times 4/\pi$ is the factor for converting square centimeters into circular mils, and 30.48 is the number of centimeters in one foot. Ans. 0.0625 mho per circular mil foot.

Prob. 2. Each field coil of an electric machine has 720 turns, the average length of a turn being about 1.5 m. What size wire is required if the hot resistance of the coil is to be 1.14 ohms? According to the A. I. E. E. Standardization Rules, a temperature rise of 50° C. is allowed above the air at 25° C. Ans. About 20 sq. mm.

Prob. 3. A given current i is to be transmitted at a given voltage between two given localities whose distance apart is l. Deduce an expression for the most economical size of the line conductor. A small conductor means a saving in the original investment, but a higher operating cost on account of the power lost in the conductor, and vice versa. The most economical conductor is one for which the annual interest and depreciation plus the cost of the i^2r loss per year is a minimum. Solution: Let the cost of one watt-year be p cents, and let the conductor cost q cents per cubic centimeter, installed. Let δ be the annual interest and depreciation in per cent to be allowed on the original cost of the conductor. The cost of the power lost in the line is $pi^2\rho l/A$, and the initial cost of the conductor is qlA. The condition of the problem is that

$$pi^2\rho l/A + \delta qlA + K = \text{min.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where the constant K represents the interest and depreciation on the poles, insulators, etc., the size of which is essentially independent of the size of the conductor. Equating the first derivative with respect to A to zero, we get $-pi^2\rho/A^2 + \delta q = 0$, or

$$pi^2\rho/A = \delta qA. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22a)$$

In other words, the most economical cross-section is that for which the sum charged to the annual interest and depreciation is equal to the cost of the wasted energy. This result is independent of the length of the line and of the voltage, and is known as *Kelvin's law of economy.*[1] Knowing all the data, the cross-section A can be calculated from condition (22a); see also problems 6 and 7 in Art. 6.

Prob. 4. A transmission line from the generating station A to a place B is l kilometers long. At B the line is divided into two branches; one to C, l_1 km. long, and carrying a current i_1; the other to D, l_2 km. long, and carrying a current i_2. The total permissible voltage drop from A to either C or D is ϵ volts (one way). Determine the sizes of the conductors in the three parts of the line so as to make the total initial cost of copper a minimum. Solution: Take the unknown voltage drop x from A to B as the independent variable; then the three cross-sections are determined by the conditions $i\rho l/A = x$, $i_1\rho l_1/A_1 = \epsilon - x$, and $i_2\rho l_2/A_2 = \epsilon - x$. The value of x itself is determined by the condition that $lA + l_1A_1 + l_2A_2 = \text{min.}$ Substituting the values of A, A_1 and A_2 into this expression, and equating the first derivative with respect to x to zero, we get $il^2/x^2 = i_1l_1^2/(\epsilon - x)^2 + i_2l_2^2/(\epsilon - x)^2$. Extracting the square root of each member of this equation and solving for x, we find that

$$x = \frac{\epsilon}{1 + [(i_1/i)(l_1/l)^2 + (i_2/i)(l_2/l)^2]^{\frac{1}{2}}}.$$

Having found x, the three cross-sections are easily calculated from the three conditions written above.

6. Current Density and Voltage Gradient.

When a current is distributed uniformly over the cross-section of a cylindrical conductor, it is convenient to speak of the *current density*, or the current per unit cross-section of the conductor. Denoting this density by U, we have

$$U = i/A. \qquad (23)$$

U is measured in amperes (or kiloamperes, milliamperes, etc.) per square centimeter, or per square millimeter. The current density is numerically equal to the current through each unit conductor of which the given conductor consists.

[1] In practice the selection of the cross-section of a line conductor is determined by many other considerations besides that of economy, as outlined above. For instance, it may be desired to reduce the original investment to a minimum while the load is small, and to change the conductors to a larger size afterwards. The problem above is intended only to introduce the reader into this subject. He will find numerous contributions treating of more complicated cases in various periodicals and transactions. See also A. C Perrine, *Electrical Conductors*, Chapter 8.

When the voltage drop is distributed uniformly along a conductor, it is convenient to speak of voltage drop per unit length. This voltage drop across a unit length is called the *voltage gradient*, and is measured in volts, kilovolts, millivolts, etc., per meter or per centimeter. Denoting the voltage gradient by G, we have

$$G = e/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The value of G characterizes the electrical condition at a point (or a cross-section) of the conductor; for this reason G is sometimes called the *electric intensity* at a point.

Having previously introduced the resistance ρ and the conductance γ of a unit conductor, we can now write Ohm's law for the unit conductor, in the form

$$G = \rho U = U/\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Equation (25) has a definite meaning also without the concept of the unit conductor; namely, it gives the relation between the voltage gradient and the current density *at a point*, for a given material. The reader can easily think of a thermal analogue. Hooke's law for elastic materials is also somewhat analogous to eq. (25), because it expresses a straight-line relation between the cause and the effect. Equation (25) can be deduced directly from eq. (1) by writing the latter in the form $Gl = (\rho l/A) \cdot UA$ and canceling l and A.

Prob. 1. What is the voltage drop per kilometer of a copper wire having a cross-section of 70 sq. mm., and carrying a current of 150 amp.? Solution: $U = 150/70 = 2.143$ amp. per sq. mm. The conductivity of copper γ, at the temperature of the line, is equal to 57 mhos for a unit conductor of one square millimeter cross-section and one meter long. Therefore, the electric intensity or the voltage drop per meter of length, according to formula (25), is $2.143/57 = 0.0376$ volt per meter.

Ans. 37.6 volts/km.

Prob. 2. What is the expression for power converted into heat in a unit conductor? Ans.

$$P = G \cdot U = U^2\rho = U^2/\gamma = \gamma \cdot G^2. \quad . \quad . \quad . \quad . \quad (25a)$$

Prob. 3. What is the amount of power lost in the conductor considered in problem 1?

Ans. $(0.0376 \times 2.143) \times 70 \times 1000 = 5640$ watts.

Prob. 4. The space available on the frame of a generator for a rectangular field coil is 16×12 cm. for the inside dimensions, and 28×24 cm. for the outside dimensions; the limiting height is 15 cm. What current density can be allowed in the coil, if 12 sq. cm. of exposed surface are required per watt loss, in order that the temperature of the coil shall

not exceed the safe limit? The space factor of the coil is 0.55; in other words, 55 per cent of the gross space is occupied by copper, the rest being taken by the air spaces and the insulation. Solution: The exposed surface is $2(28 + 24) \times 15 = 1560$ sq. cm.; therefore, 130 watts loss can be allowed in the coil. With a space factor of 0.55 the useful cross-section of copper is $6 \times 15 \times 0.55 = 49.5$ sq. cm.; the average length of one turn equals $2(22 + 18) = 80$ cm. Therefore, the coil contains $4950 \times 0.8 = 3960$ unit conductors, each one meter long and one square millimeter in cross-section. The permissible loss per unit conductor is $130/3960 = 0.0328$ watt. Hence, according to the answer to problem 2 above, $U = \sqrt{0.0328 \times 57} = 1.37$ amp. per sq. mm. This result is independent of the size of the wire, as long as the space factor remains approximately the same. The maximum ampere-turns are $137 \times 49.5 = 6780$, and, for a constant space-factor, are also independent of the size of the wire.

Prob. 5. What are the size of wire and the exciting current in the preceding problem if the voltage drop must not exceed 20 volts at 80° C.?

Ans. 4.74 sq. mm.; 6.5 amp.

Prob. 6. Referring to problem 3 in Art. 5, what is the general expression for the most economical current density? Ans. $U = \sqrt{\delta q/(\rho p)}$.

Prob. 7. Referring to the preceding problem, what is the most economical current density if copper costs 15 cents per pound, the annual interest and depreciation is taken at 12 per cent, and the estimated cost of wasted power is 22 dollars per kilowatt-year?

Ans. 0.95 amp. per sq. mm., taking ρ at 25° C.

7. Kirchhoff's Laws. Consider an arbitrary network of conductors (Fig. 4), with sources of e.m.f. connected in one or more places. When such a system is left to itself, definite currents will flow through the conductors, and definite differences of potential will be established between the junction points of the conductors. Thus, if all the resistances and e.m.fs. are given, it ought to be possible to calculate the magnitude and direction of all the currents. The distribution of the currents is such that two conditions are satisfied:

(1) As much current flows toward each junction as from it, because electricity behaves like an incompressible fluid. For any junction this is expressed mathematically by the equation

$$\Sigma i = 0, \quad \ldots \ldots \ldots \quad (26)$$

in which all the currents flowing toward the junction are taken with the sign plus, all those flowing away from it with the sign minus, or vice versa. Thus, for instance, at the point C let the currents flowing toward the junction be 20 and 30 amp. respec-

tively, and one of the currents flowing away from it be 40 amp. Then the fourth current must necessarily be 10 amp. flowing away from C, because $20 + 30 - 40 - 10 = 0$. Equation (26) is called Kirchhoff's first law.

(2) The sum of the terminal voltages along any closed circuit in the network is equal to zero, or

$$\Sigma e_t = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Consider, for instance, the path $ABCDEFA$, and connect a zero-center voltmeter first between A and B, then between B and C and so on, every time transferring both terminals, so that one

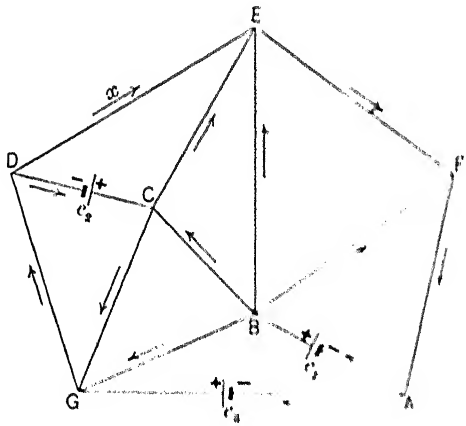

FIG. 4. A network of conductors, illustrating Kirchhoff's laws.

particular terminal of the instrument always leads the other. Consider the deflections to one side of the zero point as positive, to the other side as negative. Equation (27) means that the algebraic sum of these readings is equal to zero. The reason is as follows: The reading A-B shows how much higher is the potential at B than that at A; the reading B-C shows the amount by which the potential at C is higher than that at B. Hence the sum of the two readings indicates the difference of potential or the voltage between C and A. Consequently, the sum of all the readings around the closed circuit indicates the difference of potential between A and A, which difference is evidently zero, no matter by which closed path the original point has been reached. The reader may again resort to analogues in order to

see this law more clearly. For instance, the potentials at the joints may be likened to temperatures, and the voltmeter readings to differences of temperature. Or the potentials may be compared to absolute pressures in a network of pipes, and the voltages e_t to the differences of pressure. Again, the potentials of the points A, B, C, etc., are analogous to the altitudes of certain points, say above the sea level, while the voltages correspond to their relative elevations. In all such cases the sum of the differences around a closed path is equal to zero.

Equation (27) is usually written in a somewhat different form, because the values of e_t are usually not known, so that it is desirable to express them through the given electromotive forces and the resistances of the conductors. The general expression (4) of Ohm's law holds for each conductor in the network. Write these expressions for all the conductors along a closed path, and add them together, term by term. The sum of the e_t's is equal to zero, according to eq. (27), so that the result is

$$\Sigma e_l = \Sigma ir. \qquad (28)$$

This form of eq. (27) is known as Kirchhoff's second law. In this equation a certain direction of currents and voltages must be assumed as positive. Let, for instance, in the circuit $ABCDEFA$ the clockwise direction be taken as positive; that is, all the currents flowing clockwise are to be considered positive, and also all the e.m.fs. which tend to produce currents in the clockwise direction. Let $e_1 = 70$ volts, and $e_2 = 50$ volts, and let the resistances of the conductors be 2, 3, 5, 4, 8 and 6 ohms respectively. Let all the currents be known except that in DE, and let them be 10, 15, 15, 3 and 5 amp. respectively, the directions being those shown in the figure. Denote the unknown current in DE by x, and assume it to flow in the clockwise direction. Equation (28) then becomes

$$70 - 50 = 10 \times 2 + 15 \times 3 - 15 \times 5 + 4x + 3 \times 8 + 5 \times 6,$$

from which $x = -6$ amp. In other words, the current in DE is equal to 6 amp. and is flowing counter-clockwise.

For a given network of conductors the number of equations of the form (26) is equal to the number of junction points less one, because the equation for the last point can be obtained by combining the other equations. The number of equations of the

form (28) is equal to the number of *independent* closed paths in the network. The total number of equations of both kinds is just equal to the number of unknown currents, so that these currents can be determined by solving the simultaneous equations.

Prob. 1. A constant e.m.f. of 110 volts is maintained at the generating station, and power is transmitted through a line having a resistance of 0.5 ohm to two devices in parallel, viz., a resistor of 10 ohms, and a motor the internal resistance of which is 5 ohms. Calculate the line current (a) when the motor armature is blocked, and (b) when it revolves at such a speed that the counter-e.m.f. is 90 volts.

Ans. 28.7 amp.; 13.05 amp.

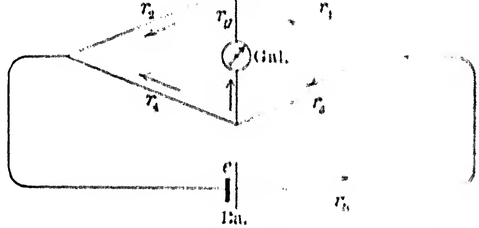

FIG. 5. An unbalanced Wheatstone bridge.

Prob. 2. Write the equations for the six unknown currents in a Wheatstone bridge (Fig. 5) when it is not balanced.

Ans. $i_b = i_2 + i_4$; $i_2 = i_1 + i_g$; $i_3 = i_4 + i_g$; $i_b r_b + i_1 r_1 + i_3 r_3 = e$; $i_1 r_1 - i_3 r_3 - i_g r_g = 0$; $i_2 r_2 - i_4 r_4 + i_g r_g = 0$.

Prob. 3. Show that the preceding six equations are reduced to three when the galvanometer circuit is open.

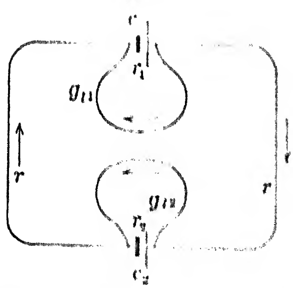

FIG. 6. A leaky electric circuit containing a counter-e.m.f.; this is an analogue to the magnetic circuit of a loaded electric machine.

Prob. 4. Two sources of e.m.f., e_1 and e_2 (Fig. 6), are connected to act against each other, e_1 being larger than e_2. The internal resistances of these sources are r_1 and r_2 respectively; the main external resistance is $2r$. The insulation between the terminals of the sources of e.m.f. is

imperfect, and the leakage conductances are represented by g_{l1} and g_{l2} respectively. Write Kirchhoff's equations for the unknown currents (a) when there is no leakage; (b) when $g_{l2} = 0$; (c) when $g_{l1} = 0$; and (d) when both leakages are present.[1]

Prob. 5. A telegraph line with ground return has a resistance of r' ohms per kilometer, and a leakage conductance to the ground of g' mhos per kilometer.[2] The voltage at the receiver end of the line is E_2, the receiver current I_2. What are the values of the voltage and of the current at a distance of s kilometer from the receiving station? Solution: Consider an infinitesimal length ds of the line, at a distance s from the receiving end, and let the voltage to the ground at this point be e. If i is the line current at the same point, then the leakage current corresponding to the element ds of the line is di, and we have, according to Ohm's law, $di = eg'ds$, where $g'ds$ is the leakage conductance through the element ds of the line. For the element of the line itself, Ohm's law gives $de = ir'ds$. Substituting the value of e from the first equation into the second, gives $d^2i/ds^2 = r'g'i$, or i is such a function of s that its second derivative is proportional to the function itself. The solution of this differential equation is $i = A_1\epsilon^{-ms} + A_2\epsilon^{+ms}$, where $m = \sqrt{r'g'}$, and A_1 and A_2 are the constants of integration. However, in our case it is preferable to express the solution through hyperbolic functions, in the form $i = C_1 \cosh ms + C_2 \sinh ms$, where $m = \sqrt{r'g'}$ and C_1 and C_2 are the constants of integration. The reader can check this solution by substituting it in the differential equation. The constants of integration are determined from the given conditions at the receiver end of the line. Namely, from $di = eg'ds$ we find $e = (1/g')(di/ds) = (m/g')(C_1 \sinh ms + C_2 \cosh ms)$. For $s = 0$, $e = E_2$ and $i = I_2$. Consequently, $C_1 = I_2$; $C_2 = E_2g'/m$.

Prob. 6. Referring to the preceding problem, the resistance of a telegraph line is 7 ohms per kilometer, and the insulation resistance to the ground is 1.2 megohms per kilometer; the line is 400 kilometers long. A relay of 300 ohms resistance and requiring 0.12 ampere to operate it, is connected between the receiver end of the line and the ground. Calculate the required current and battery voltage at the sending station.

Ans. 0.195 amp.; 445 volts.

[1] The electric circuit shown in Fig. 6 is of importance because it serves as a good analogue to the magnetic circuit in a loaded machine. The electromotive forces e_1 and e_2 correspond to the magnetomotive forces of the field and the armature respectively, the reluctances of the parts of the main path being represented by $2r$, r_1 and r_2, while the leakage permeances correspond to g_{l1} and g_{l2}. See the author's *Magnetic Circuit*, the latter part of Art. 40, and problem 13.

[2] Primed symbols are used in this book and in the *Magnetic Circuit* where quantities refer to unit length.

CHAPTER III

CONDUCTORS OF VARIABLE CROSS-SECTION[1]

8. Current Density and Voltage Gradient at a Point. When the cross-section of a conductor varies along its length (Fig. 7), the voltage drop per unit length and the current density are also variable. In places like MN, where the cross-section of the conductor is comparatively small, the resistance per unit length is correspondingly large, and vice versa. Consequently, the voltage gradient and the current density are also larger at MN than, for example, at PQ. Equations (23) and (24) give in this case only an *average* current density and an average voltage gradient over the conductor.

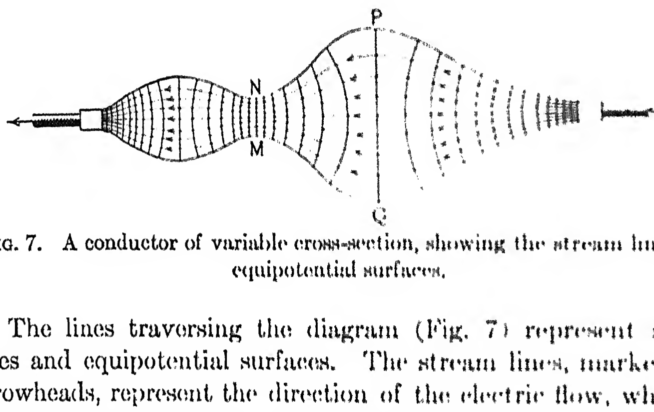

FIG. 7. A conductor of variable cross-section, showing the stream lines and equipotential surfaces.

The lines traversing the diagram (Fig. 7) represent stream lines and equipotential surfaces. The stream lines, marked with arrowheads, represent the direction of the electric flow, while the equipotential surfaces are perpendicular to them, and are the loci of points of equal potential. The distribution is analogous to that

[1] This chapter may be omitted if desired, because it is not necessary for the understanding of the following chapters on alternating currents. The importance of this chapter lies in the fact that the treatment is analogous to that of the electrostatic circuit, and therefore it greatly facilitates the study of the latter. This chapter may, therefore, be conveniently studied before taking up Chapter 14. The treatment is also analogous to that used in the author's *Magnetic Circuit*.

which obtains in the flow of heat; the stream lines indicate the direction of the flow of heat, while the equipotential surfaces are analogous to those of equal temperature.

In order to understand the meaning of equipotential surfaces, let one lead of a voltmeter be applied at one of the terminals of the conductor, and let the other lead be moved about inside the conductor (assuming this to be possible), marking the points for which the deflection of the voltmeter remains the same. All the points for which the reading is, let us say, 10 volts form an equipotential surface; while all those for which the voltmeter reads 11 volts form another equipotential surface, and so on. Between two points on the same equipotential surface the voltmeter reading is evidently zero. The equipotential surfaces are perpendicular to the stream lines, because if there were a component of flow along an equipotential surface there would be an *ir* drop between two points on the same surface, and the voltage between these two points could not be zero.

Stream lines and equipotential surfaces give a clear idea of the character of flow of a current in a conductor of irregular shape, especially if they are drawn to correspond to equal increments of current and voltage. This means that the lines of flow should be drawn so as to define *tubes of current* of equal strength. For instance, in Fig. 7 the current included between any two adjacent stream lines is supposed to be the same—let us say, equal to one ampere. Similarly, the voltage between any two adjacent equipotential surfaces should be the same; for example, one volt. If the lines are drawn sufficiently close together, they give complete information about the voltage and current relations in the different parts of the conductor, and also show places of high and low current density and voltage gradient.

The true current density *at a point* is obtained by considering an infinitesimal tube of current di and dividing di by the infinitesimal cross-section dA of the tube at the point under consideration. Then, instead of eq. (23), we have

$$U = di/dA. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

If, on the other hand, it is desired to express the total current through the density, the preceding relation gives

$$i = \int_0^A U dA, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

the integration to be extended over the whole equipotential surface, U being a function of the position of dA. In other words, *current is the surface integral of current density.*

The relation between the variable intensity G along the conductor, and the total voltage e at its terminals, is no longer expressed by the simple relation (24), applicable to the whole conductor. Relation (24) must now be written for an infinitesimal length dl of a stream line, because G is constant only for an infinitesimal length. The definition of G remains the same, namely, G is the rate of variation of voltage per unit length of the conductor. Thus, denoting by de the voltage between two adjacent equipotential surfaces at a distance dl apart, we have

$$G = de/dl, \quad \text{or} \quad de = G \cdot dl. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The total voltage e between the terminals of the conductor is equal to the sum of these infinitesimal drops, or

$$e = \int_0^l G \cdot dl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Equation (32) is expressed in words by saying that *voltage is the line integral of electric intensity* (or voltage gradient).

A clear understanding of relations (31) and (32) is of paramount importance in the study of electrostatic and magnetic phenomena. This will be aided by recalling to mind the thermal analogy previously used. In the case of the flow of heat, G corresponds to the rate of change in temperature per unit length of the rod, while e represents the total difference of temperature between the ends of the rod. Equation (31) expresses the fact that, by taking the rate at a certain point and multiplying it by a very short element of the length of the rod, the actual difference of temperature between the ends of this element is obtained. Thus, for instance, let the drop in temperature at some point of the rod be equal to 2.5° C. per meter length. Then the actual drop in a very short element, say 0.1 mm., is $2.5 \times 0.0001 = 0.00025°$ C. The element of length must be small, because by supposition the cross-section of the rod is not constant, and the rate of drop is consequently variable. For a short length the variable quantities can be assumed constant, or, more correctly, average values can be used. Equation (32) thus states that the total difference of temperature between the ends of the rod is equal to the sum (or the integral) of the drops in the very small elements

Similarly, in a pipe of variable cross-section the rate of loss of head per unit length is variable, so that it is only possible to speak of this rate G at a point. The total loss of pressure, or head e, is obtained by summing up the small losses of head in infinitesimal elements of the pipe. The loss of pressure for a length dl is $G\,dl$; the total head e is the integral of this expression over the whole length of the pipe. This is expressed mathematically by eq. (32).

Relation (25) between G and U holds true for a non-uniform flow as well, because it merely gives a relation between the cause G and the effect U *at a point*, depending only upon the property of the material, as expressed by the factor γ or ρ. This relation may be also considered as Ohm's law for an infinitesimal cylindrical conductor of length dl and cross-section dA, namely,

$$G\,dl = de = (\rho\,dl/dA)U\,dA.$$

Canceling dA and dl, relation (25) is obtained.

Prob. 1. A current of 50 amp. is flowing along a cylindrical conductor 3 cm. in diameter. The resistivity of the material varies in concentric layers in such a way that the current density is proportional to the cube of the distance from the axis. What is the current density at the periphery? Ans. 17.7 amp. per sq. cm.

Prob. 2. A conductor of circular cross-section, 225 cm. long, has the form of a truncated cone, the diameters of the two terminal cross-sections being 1.2 cm. and 3 cm. respectively. The total drop at a certain current is 65 volts. What is the general expression for the voltage gradient G_x at a distance x from the smaller end?

Ans. $G_x/G_0 = [a/(a+x)]^2$, where $a = 150$ cm. is the distance from the smaller end to the apex of the cone, and $G_0 = 0.723$ volt per centimeter is the voltage gradient at the smaller end. G_0 is determined from eq. (32), namely, $65 = G_0 \int_0^{225} a^2\,dx/(a+x)^2$.

Prob. 3. A non-linear irregular conductor, made of homogeneous material, has a current density U and an electric intensity G, varying from point to point in magnitude and direction. What is the general expression for the power converted into heat?

Ans. According to eq. (25a),

$$P = \int G \cdot U\,dv = \frac{1}{\gamma}\int U^2\,dv = \gamma \int G^2\,dv, \quad . \quad . \quad . \quad (33)$$

where dv is the element of volume to which G and U refer, and the integration is extended over the whole volume of the conductor. The volume dv must be taken as a cylinder or parallelopiped, the length of which is in the direction of flow of the current, the cross-section being perpendicular to this flow.

9. The Radial Flow of Current. The solution of problems involving a non-uniform flow of current usually requires considerable facility in the use of higher mathematics beyond ordinary calculus. An exception to this statement is the simple case of radial flow (Fig. 8) between two concentric electrodes, cylindrical or spherical. The following exercises give an opportunity for practice in the solution of problems of this kind. They serve to illustrate the concepts of current density and voltage gradient, and to prepare the student's mind for the solution of certain problems on concentric cables, involving the dielectric and magnetic circuits.

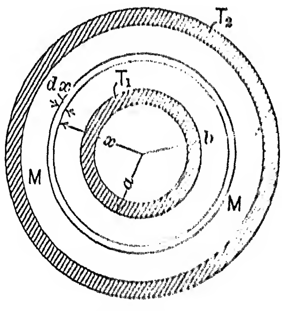

FIG. 8. Flow of current between two concentric electrodes.

Prob. 1. Calculate the resistance of a cylindrical layer of mercury MM (Fig. 8) of height $h = 5$ cm., between two concentric cylindrical terminals T_1 and T_2, the radii of the contact surfaces being $a = 10$ cm. and $b = 18$ cm. The resistivity of mercury is 95 microhms per cubic centimeter. Solution: Take an infinitesimal layer of the mercury, between the radii x and $x + dx$; the resistance of this layer is $\rho \cdot dx/(2\pi xh)$. The resistances of all the infinitesimal concentric layers are in series; therefore, r is obtained by integrating the foregoing expression between the limits a and b, the result being

$$r = [\rho/(2\pi h)] \cdot \operatorname{Ln}(b/a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Ans. $r = 1.775$ microhms.

Prob. 2. In the preceding problem, when a current of 10,000 amp. flows through the mercury, what is the amount of heat generated per second per cubic centimeter of mercury, at both electrodes? Solution: The current density at the inner electrode is $10{,}000/(2\pi \times 10 \times 5) = 31.8$ amp. per square centimeter. According to eq. (25a), the loss of power is $(31.8)^2 \times 95 \times 10^{-6} = 0.0958$ watt per cubic centimeter. The heat loss at the outer electrode is $0.096 \times (10/18)^2 = 0.0295$ watt per cubic centimeter.

Prob. 3. What is the curve of electric intensity G as a function of x in the preceding two problems, and what are the limiting values of G?

Ans. An equilateral hyperbola; $G_1 = 3.02$, $G_2 = 1.677$ millivolts per centimeter length.

Prob. 4. A lead-covered cable, consisting of a solid circular conductor of A square millimeters in cross-section, is insulated with a layer of rubber c mm. thick, between the conductor and the sheathing. What is

the insulation resistance of l kilometers of such a cable, if the resistivity of rubber is ρ megohms per centimeter cube?

Ans. $[\rho \times 10^{-5}/(2\pi l)] \cdot \text{Ln}\,(1 + 1.772\, c/\sqrt{A})$ megohms, according to eq. (34).

Prob. 5. Show that by doubling the thickness of the insulation in the preceding problem, the insulation resistance is increased less than twice.

Prob. 6. A current is flowing through a hemispherical shell of metal along radial lines. Express the resistance of the shell as a function of its radii a and b, and the conductivity γ of the material.

Ans. $(b - a)/(2\pi\gamma ab)$.

Prob. 7. Apply the method of superposition and the result obtained in Arts. 60 and 63, to the calculation of the resistance of an unlimited conducting medium between two parallel cylindrical terminals. Such a case obtains, for instance, when a load resistor consists of two vertical pipes in a pond, the pipes being used as the terminals, the current flowing through the water.

10. The Resistance and Conductance of Irregular Paths. Let a conductor of irregular shape (Fig. 7) be connected to a source of constant voltage e. The power converted into heat in the conductor is e^2/r, where r is the resistance of the conductor. This resistance depends upon the distribution of the current in the body of the conductor. The general law, demonstrated by all experiments, is that the distribution of the current is such as to make the dissipated energy a maximum. Since by supposition e is constant (unlimited supply), the resistance r must be a minimum.

Let now the same conductor be connected to a source of constant current — for instance, an arc-light machine. The distribution of the current in the conductor is such as to effect its passage with a minimum expenditure of energy, that is, minimum voltage at the terminals, or minimum i^2r. This again means that the resistance r is a minimum. The student is advised to consider similar cases in the flow of heat or of a fluid, in order to make the matter perfectly clear to himself.

The general law of nature — that of minimum effort or minimum resistance — applies in all such cases, and is used in the calculation of the resistance of conductors of irregular form. The conductor is divided into small parts by means of stream lines and equipotential surfaces as shown in Fig. 7, drawing them to the best of one's judgment. These small cells are nearly cylindrical in form, so that their resistances or conductances are easily esti-

mated by using their mean lengths and average cross-sections. The resistance of the whole conductor is found by properly combining the resistances of these cells in series, and the conductances of the filaments thus obtained in parallel. Then the assumed shapes of the stream lines and of the equipotential surfaces are somewhat modified, and the resistance is calculated again, and so on. Thus, by successive trials, the minimum resistance, or the maximum conductance, of the given conductor is found, and this is the true value of resistance or conductance, as the case may be. The lines corresponding to this minimum give the true distribution of currents and voltages within the conductor.

The work of the trials is made more systematic by following a procedure suggested by Lord Rayleigh, and further developed by Dr. Lehmann. This method is described in detail in Art. 54 below, in application to the electrostatic field, and also in Art. 41 of the author's *Magnetic Circuit*, in application to the magnetic field. The student will have no difficulty in applying the method to an electro-conducting circuit. The best way to make it clear to one's self is actually to draw a conductor of irregular shape (in two dimensions for the sake of simplicity) and to calculate its resistance in the above-mentioned manner.[1]

11. The Law of Current Refraction. The method outlined above for the mapping out of stream lines and equipotential surfaces applies only in a homogeneous conductor. When a current passes from one substance to another (Fig. 9), the stream lines suddenly change their direction at the dividing surface AB between the media, and in so doing they obey the law of current refraction, which is

$$\tan\theta_1/\tan\theta_2 = \gamma_1/\gamma_2. \qquad (35)$$

Here θ_1 and θ_2 are the angles of incidence and refraction, while γ_1 and γ_2 are the respective conductivities of the two media. This equation shows that the lower the conductivity of a substance, the more nearly do the stream lines approach the direc-

[1] In two-dimensional problems of this kind, the properties of conjugate functions may be used when the geometric forms involved can be expressed by analytic equations. However, the purely mathematical difficulties are such as to make this method applicable only in a comparatively few simple cases. See J. C. Maxwell, *Electricity and Magnetism*, Vol. 1, p. 284; J. J. Thomson, *Recent Researches in Electricity and Magnetism*, chap. 3; Horace Lamb, *Hydrodynamics*, chap. 4.

tion of the normal N_1N_2 at the dividing surface. In this way, the path between two given points is shortened in the medium of lower, and is lengthened in that of higher conductivity, by such an amount in each case that the total conductance of the composite conductor is larger with refraction than without it. Hence, the existence of refraction is a necessary consequence of the general law of least resistance.

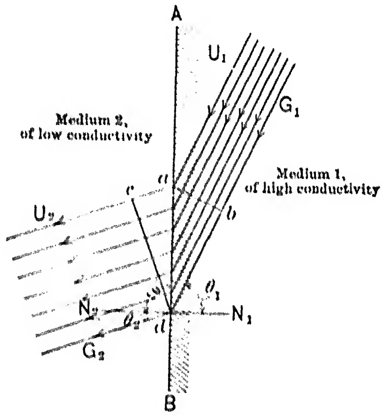

FIG. 9. The refraction of a current, or of a flux.

To deduce eq. (35), consider a tube of current between the equipotential surfaces *ab* and *cd*, and let the width of the path in the direction perpendicular to the plane of the paper be one centimeter. Let U_1 and U_2 be the current densities in the tube, and let G_1 and G_2 be the corresponding voltage gradients. Two conditions must be satisfied, namely, (1) the total current through *cd* is equal to that through *ab*, and (2) the voltage drop along *ac* is the same as that along *bd*. These conditions are expressed by the equations

$$U_1 \cdot \overline{ab} = U_2 \cdot \overline{cd}$$

and

$$G_1 \cdot \overline{bd} = G_2 \cdot \overline{ac}.$$

Dividing the first equation by the second and rearranging the terms gives

$$\frac{U_1/G_1}{\overline{bd}/\overline{ab}} = \frac{U_2/G_2}{\overline{ac}/\overline{cd}}.$$

But, according to eq. (25), $U_1/G_1 = \gamma_1$, and $U_2/G_2 = \gamma_2$. From Fig. 9, $\overline{bd}/\overline{ab} = \tan \theta_1$, and $\overline{ac}/\overline{cd} = \tan \theta_2$. By substituting these values in the preceding equation, relation (35) is obtained.

Thus, in mapping out an electro-conducting circuit in two media, the stream lines must be so drawn as to satisfy eq. (35), and the conductance must be a maximum for the combination, and not for each part separately. A similar law applies to electrostatic and magnetic fluxes (see Art. 55 below, and Art. 41a of the author's *Magnetic Circuit*).

Prob. 1. Make clear to yourself the reason why the refraction of light follows a sine law, while in the case of the electric current it is a law of tangents.

Prob. 2. Show that total reflection is impossible for an electric current.

Prob. 3. Draw a set of curves giving values of θ_1 for different values of θ_2 when the ratio of conductivities is 1, 2, 10 and 100.

CHAPTER IV

REPRESENTATION OF ALTERNATING CURRENTS AND VOLTAGES BY SINE-WAVES AND BY VECTORS

12. Sinusoidal Voltages and Currents. A large proportion of the electric power used for lighting, industrial purposes, and traction is generated in the form of alternating currents. Some of the advantages of the alternating current over the direct current are: (1) Alternating-current power can be easily converted into power at a higher or at a lower voltage, thus making possible the transmission of power over long distances; (2) the generation of alternating currents is simpler than that of direct currents, the latter requiring a commutator,[1] which needs constant attention in operation; and (3), by combining two or three alternating-current circuits into a *polyphase* system it is possible to convert electric into mechanical power, using motors of simple and rugged construction (induction motors and synchronous motors).

Alternating voltage waves generated by commercial alternators are more or less irregular in shape, but for most engineering calculations it is accurate enough to assume them to vary with the time according to the sine law (Fig. 10). This assumption simplifies the theory and calculations greatly; moreover, the results obtained with this assumption are comparable with one another, because they all refer to a standard shape of the voltage and current curves, instead of a particular form in each specific problem. If the curve of a voltage or current differs greatly from the sine-wave, it can be resolved into a series of sine-waves of different frequencies, so that even then the sine-wave remains the fundamental form (see Art. 15 below). Fig. 10 shows the well-known construction of a sine-wave, the instantaneous values of the current or voltage being represented as

[1] The homopolar machine, which is a direct-current machine without a commutator, has not proven, up to the present time, to be commercially successful.

ordinates, against time as abscissæ. Instead of actual time in seconds, the curve is sometimes plotted against some other quantity proportional to time—for instance, fractions of a complete cycle. It is sometimes convenient to use as abscissæ the angular positions of a field pole of the alternator with respect to an armature conductor in which the electromotive force under consideration is induced.

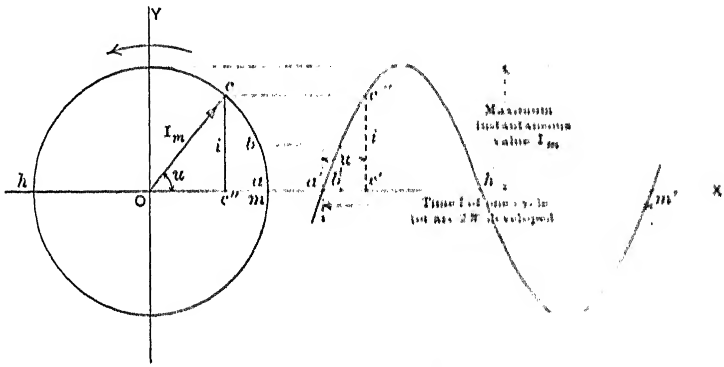

FIG. 10. An alternating current represented by a sine-wave.

To construct the curve of an alternating current or voltage, draw a circle the radius of which equals the maximum value of the wave. Divide the circle into a certain number of equal or unequal parts, such as ab, bc, etc., and mark on the axis of abscissæ points a', b', c', etc., corresponding to the points of division on the circle. That is, $a'b'$ is either equal or proportional to ab; $b'c'$ is either equal or proportional to bc, and so on. In general, an abscissa such as $a'c'$ represents, to a certain selected scale, the central angle u corresponding to the arc ac. The length $a'm'$ represents to the same scale an angle of 360 degrees, or the time of one complete cycle of the wave. The ordinates of the sine-wave are equal to the corresponding ordinates of the circle. For example, the point c''' on the curve is obtained by transferring the ordinate cc'' of the circle to the corresponding abscissa $a'c'$.

The name "sine-wave" is derived from the fact that these ordinates are proportional to the sines of the abscissæ, which represent to some scale the central angles of the circle of reference. The equation of the curve expresses this property analytically. Let the maximum value of the current, which is also

equal to the radius of the circle, be denoted by I_m; we have then from the triangle Occ''

$$i = I_m \sin u, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where the ordinate $i = cc'' = c'c'''$ represents the instantaneous value of the alternating current, at the moment of time corresponding to the angle u. The variable angle u is proportional to the time, because the radius Oc which generates the sine-wave is assumed to revolve at a uniform speed. Let time t be counted from the position Oa of this radius, and let $T = a'm'$ be the interval of time necessary to complete one revolution of the radius, or the time of one complete *cycle* of the alternating wave. When $t = 0$, $u = 0$; and when $t = T$, $u = 2\pi$. Therefore, in general,

$$u = 2\pi t/T, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

because this expression satisfies the foregoing conditions. Substituting this value of u into eq. (36), we obtain

$$i = I_m \sin (2\pi t/T). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

For the values of $t = 0$, $\frac{1}{2} T$, T, $\frac{3}{2} T$, etc., $i = 0$, as one would expect, because at these moments the current changes from positive to negative values, or vice versa. At $t = \frac{1}{4} T$, $\frac{3}{4} T$, $\frac{5}{4} T$, etc., we have $i = \pm I_m$; at these moments the current reaches its positive and negative maxima. Equation (36) is used when the sine-wave is plotted against the values of angle as abscissæ. Equation (38) gives the same curve referred to time as abscissæ.

In practice, the rapidity with which currents and voltages alternate is not denoted by the fraction of a second T during which a cycle is completed, but, in a more convenient manner, by the number of cycles per second. Thus, instead of saying that an alternator generates current which completes a cycle within $\frac{1}{60}$ of a second, it is customary to say that the frequency of the current is 60 cycles per second. Denoting the frequency in cycles per second by f, we have

$$f = 1/T, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and consequently

$$i = I_m \sin 2\pi ft. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

This is the usual expression for an alternating current having a frequency of f periods per second. Analogously, for an alternating voltage we have

$$e = E_m \sin 2\pi ft, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where E_m is the maximum instantaneous value, also called the *amplitude*, and e is the instantaneous value of the voltage at the time t.

In numerical calculations, and when drawing sine-waves, the values of the ordinates for various values of u or t are obtained either graphically, as in Fig. 10, or from a table of sines. For approximate calculations, values of sines can be taken from a slide-rule. In the problems which follow, the student is advised to become familiar with each of the three methods of obtaining values of sines.

In some cases one has to deal with two currents or voltages of the same frequency—for instance, in two different parts of the same circuit. The two corresponding sine-waves (Fig. 11) usu-

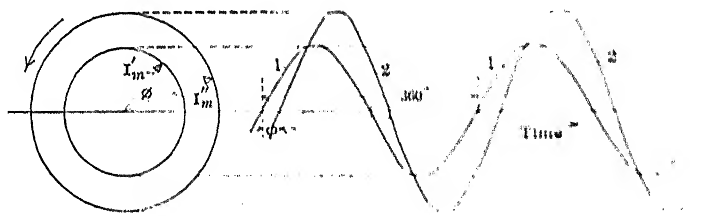

FIG. 11. Two alternating currents displaced in phase by an angle ϕ.

ally differ in amplitude, and also pass through zero at different instants. Thus, in Fig. 11, when the current 1 is at a maximum the current 2 is still growing, and passes through its maximum somewhat later. In other words, the current 2 *lags* behind the current 1, or, what is the same, the current 1 *leads* the current 2. The angle ϕ between the zero points (or between the maxima) of the two waves is called the angle of phase difference, or simply the *phase angle*. If we regard the two waves as being formed by the revolving radii I_m' and I_m'', then ϕ is the angle between the radii at any instant.

When the two waves are of different frequencies, there is no constant phase angle between them; but this angle varies periodically, so that at some intervals of time the two waves are nearly in phase, and at others they are nearly in opposition. The familiar method of synchronizing two alternators by means of incandescent lamps is based on this phenomenon.

Prob. 1. An alternating current fluctuates according to the sine law between the values of ± 75 amp., making 6000 alternations per minute (3000 positive and as many negative ones). Draw a curve of instantaneous values of this current; mark on the axis of abscissæ the time t in thousandths of a second, the angles u in degrees, and the same angles in radians.

Prob. 2. What is the frequency of the current in the preceding problem, in cycles per second? Ans. 50.

Prob. 3. Plot on the same curve sheet with the curve obtained in problem 1 the sine-wave of a current the frequency of which is three times as great, and the amplitude, 52 amp. The curve is to be at its maximum when the first curve is at a maximum.

Prob. 4. Supplement the preceding curves by one, the frequency of which is 50 cycles per second, the amplitude 63 amp., and which reaches its positive maxima at the same instants in which the first curve passes through zero. Show that with these data two distinct curves can be drawn.

Prob. 5. Draw on the same curve-sheet with the preceding curves a sine-wave representing a 50-cycle alternating current, the amplitude of which is 120 amp., and which lags by 30 degrees with respect to the current in problem 1.

Prob. 6. The current mentioned in problem 1 is generated by a 12-pole alternator, that in problem 3 by a 14-pole machine. At what speeds must these machines be driven in order to give the required frequencies? Ans. 500 and 1285 r.p.m.

Prob. 7. Express the currents given in problems 1 to 5 by equations of the form of eq. (36). Ans. $i = 75 \sin u$; $i = -52 \sin 3u$; $i = \pm 63 \cos u$; $i = 120 \sin (u - 30^{\circ})$.

Prob. 8. The angle u in the answers to the preceding problem is expressed in degrees; rewrite the equations so as to have u expressed in radians, and in fractions of a cycle. Also represent the currents as functions of the time t.

Prob. 9. Express by equations similar to eq. (41) the following sinusoidal voltages of frequency f: (a) Amplitude E_m volts. (b) Amplitude E_m' volts, lagging α degrees with respect to the first curve. (c) Amplitude E_m'' volts, leading the second curve by ϕ radians. (d) Amplitude E_m''' volts, lagging one nth of a cycle with respect to the curve (a).

Prob. 10. The voltages required in the preceding problem are induced by four identical alternators, having p poles each, and coupled together. By what geometrical angles must the revolving or the stationary parts be displaced in order to give the required differences in phase?

13. Representation of a Sine-wave by a Vector. It is clear from the foregoing theory and problems that all sine-wave currents or voltages are different from one another in three respects only, namely: (1) in amplitude; (2) in frequency; and (3) in relative phase position. In most practical cases, all the currents

and voltages entering into a problem are of the same frequency so that they differ from each other solely in their amplitudes anc phase positions. In such cases it is not necessary to draw sine waves, or even to write their equations; it is sufficient to indicate the radii I_m' and I_m'' which generate these curves (Fig. 11), in thei true magnitudes and *relative* positions. The rotating radius, a any instant, gives by its vertical projection to scale the magnitud of the alternating current or voltage at that instant.

The *absolute* position of the radii is immaterial, because they are revolving all the time. It is their relative position which i permanent, and which determines the relative position of th sine-waves. The moment from which time is counted is arbi trary in most problems; hence, one of the radii can be drawn i any desired position. Then, all other radii in the same probler are determined by their phase displacement with respect to thi "reference" radius.

It must be clearly understood that the foregoing representa tion by vectors is true only when all the vectors are revolving a the same speed, that is, only with alternating quantities of th same frequency. When currents and voltages of different fre quencies enter into a problem, the angle between the vector varies all the time, and it is necessary to introduce an arbitrar zero of time for reference. In general, the graphical method c solution is unsuitable for such problems.

In mathematics and physics, a quantity which has not only magnitude, but also a definite direction in space or in a plane is called a *vector*. Thus, for instance, in mechanics, force is vector quantity, while volume is not. The radii which repre sent sine-waves have both magnitude and direction in a plane It is proper, therefore, to call them vectors. While the directio of the first vector is usually arbitrary, once it is selected, th directions of all the other radii become definite, so that wit this limitation, the radii in alternating-current problems hav definite directions and may be called vectors. While they mus be imagined as revolving when generating their respective sine waves, yet they revolve as a system, maintaining their relativ positions unchanged. The required relations always depen upon the relative positions of the radii, so that the fact that the are revolving can be altogether disregarded, and the radii consid ered as simple stationary vectors.

Prob. 1. Draw the vectors of the currents in problems 1, 4 and 5 of the preceding article in their true magnitudes and relative positions.

Prob. 2. A single-phase alternator has a terminal voltage of which the maximum instantaneous value is equal to 16 kilovolts. The maximum value of the current supplied by the machine is 325 amp. The character of the load is such that the current wave lags behind the voltage wave by an angle of 37 degrees. Assuming both the voltage and the current to vary according to the sine law, represent the foregoing conditions by two vectors.

Prob. 3. Draw a vector diagram showing the phase (star) voltages and currents of a 25-cycle three-phase system (Fig. 36), the amplitude of each voltage being 7235 volts, and each displaced in phase by 120 degrees with respect to the other two voltages. The current in the first phase is 30 amp., and lags behind the corresponding phase voltage by $\frac{1}{8}$ of a cycle. The current in the second phase is 47 amp., and leads its voltage by 18 degrees. The current in the third phase is 72 amp., and lags behind the corresponding phase voltage by 0.004 of a second.

Note: In the foregoing three problems the student is supposed to draw the vectors equal in length to the amplitudes of the alternating waves. In practice, it is customary to draw vectors equal in length to the *effective* values of voltages and currents, and not to their amplitudes. For sine-waves the effective value is equal to the amplitude divided by $\sqrt{2}$ (see Chapter 5). The difference is not important for our present purposes. The use of effective values would merely change the arbitrary scale to which the vectors are drawn.

14. Addition and Subtraction of Vectors. There are many practical problems in which alternating currents or voltages have to be added, or subtracted one from another. For instance, when two or more alternators are working in parallel, the total current delivered to the station bus-bars is equal to the sum of the currents supplied by each machine. Or, to find the voltage at the receiving end of a transmission line, the voltage drop in the line is subtracted from the generator voltage. When the component quantities vary according to the sine law and are all of one frequency, the resultant quantity is also a sine curve of the same frequency. This curve may be found (a) graphically, by adding the component curves point by point; (b) analytically, by adding their equations; or (c) by adding the vectors of these curves.

It must first be proved that the sum of two sine-waves of one frequency is also a sine-wave of the same frequency. Let the two currents to be added be represented by the equations

$$\left.\begin{aligned} i &= I_m \sin(u+\phi) \\ i' &= I_m' \sin(u+\phi') \end{aligned}\right\}, \qquad (42)$$

where $u = 2\pi ft$ is the variable time angle, and ϕ and ϕ' are two constant angles characterizing the relative phase positions of the two waves with respect to some reference wave $I_m'' \sin u$. The phase displacement between i and i' is $\phi' - \phi$. Expanding the foregoing sines of the sum of two angles, and adding the two equations, member for member, we obtain

$$i_{eq} = i + i' = (I_m \cos\phi + I_m' \cos\phi') \sin u + (I_m \sin\phi + I_m' \sin\phi') \cos u, \quad . \quad . \quad . \quad . \quad (43)$$

where the constant coefficients of $\sin u$ and $\cos u$ are grouped together. The subscript eq stands for "equivalent." This expression is of the form $i_{eq} = A \sin u + B \cos u$, where A and B are constants. No matter what values A and B may have, the right hand side of this equation is reducible to the form

$$i_{eq} = I_{eqm} \sin(u + \phi_{eq}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Assuming this equation to be true, we equate the right-hand sides of eqs. (43) and (44), and expand $\sin(u + \phi_{eq})$. Equating the coefficients of $\sin u$ and $\cos u$, we get

$$\left.\begin{aligned} I_{eqm} \cos\phi_{eq} &= I_m \cos\phi + I_m' \cos\phi'; \\ I_{eqm} \sin\phi_{eq} &= I_m \sin\phi + I_m' \sin\phi'. \end{aligned}\right\} \quad . \quad . \quad . \quad (45)$$

These are two simultaneous equations with I_{eqm} and ϕ_{eq} as the unknown quantities. Squaring and adding these equations, we obtain

$$I_{eqm}^2 = (I_m \sin\phi + I_m' \sin\phi')^2 + (I_m \cos\phi + I_m' \cos\phi')^2. \quad (46)$$

Dividing the second equation by the first gives

$$\tan\phi_{eq} = (I_m \sin\phi + I_m' \sin\phi') / (I_m \cos\phi + I_m' \cos\phi'). \quad (47)$$

No matter what values I_m, I_m', ϕ and ϕ' may have, the values of I_{eqm} and ϕ_{eq} determined from these equations are *real*. In other words, it is always possible to represent eq. (43) in the form of eq. (44). This proves the proposition, because we see from eq. (44) that i_{eq} is a sine-wave having the same $u = 2\pi ft$ for the variable angle, hence the same frequency as the component waves. The amplitude and the phase position of this resultant wave are determined by eqs. (46) and (47).

When two currents or voltages are represented by vectors, their sum or difference is also a vector, because, as proved before,

it is also a sine-wave of the same frequency. The problem is to find the vector of the resultant wave, knowing the vectors of the component waves in their magnitudes and positions. Any ordinate of the resultant wave must be equal to the sum of the corresponding ordinates of the component waves. Hence, the vector of the resultant wave must satisfy the condition that its projection upon the Y-axis (Fig. 12) shall be equal to the sum of the projections of the component vectors on the same axis. This condition must be fulfilled at all instants of time, that is, during the rotation of the three vectors. To satisfy this requirement the resultant vector must be the diagonal of a parallelogram of which the other two vectors are the adjacent sides.

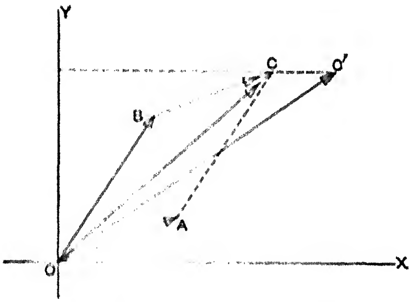

FIG. 12. Addition of vectors.

Let OA and OB be the given vectors to be added together. From the end B of the vector OB draw a line BC equal and parallel to OA. Connecting O and C gives the resultant vector OC, in magnitude and position. It will be seen from the figure that the projection of OC upon the Y-axis is equal to the sum of the projections of OB and BC upon the same axis. But BC is equal and parallel to OA, so that the projection of OC on the vertical axis is equal to the sum of the projections of the given vectors on the same axis. This construction holds true for any instant whatever. By drawing AC, the parallelogram $OBCA$ is completed, so that the construction is identical with that for finding the resultant of two mechanical forces. However, in practical applications it is not necessary to complete the parallelogram, because the resultant is perfectly determined by the triangle OBC. The resultant of two vectors obtained in this way is called their *geometric sum*.

If the triangle were not closed, the condition of equality with the sum of the projections of the given vectors might be satisfied for one particular instant of the cycle, but would not be satisfied for other instants. Thus, for instance, assuming the line OC'' to be the resultant vector, we see that for the instant shown in the sketch the projection of OC'' upon the axis OY is equal to the sum of the projections of OB and BC' upon the same axis; but the condition is not fulfilled when the vectors rotate.

The rule for subtraction of vectors follows immediately from the preceding rule, because to subtract a vector means to add a vector with the opposite sign. Thus, let it be required to subtract the vector OA from OB (Fig. 13); this may mean, for instance, the subtraction of the voltage wave represented by OA from that represented by OB. From the end B of OB draw vector BC' equal and *opposite* to OA. The resultant, OC', represents the difference of the two given vectors, in direction and magnitude, and thus determines the sine-wave of the resultant voltage. If it were required to subtract OB from OA, it would be necessary to draw AC'' equal and opposite to OB, thus obtaining the resultant OC'', equal and opposite to the former resultant OC'. This is in accord with the general algebraic rule that $A - B = -(B - A)$.

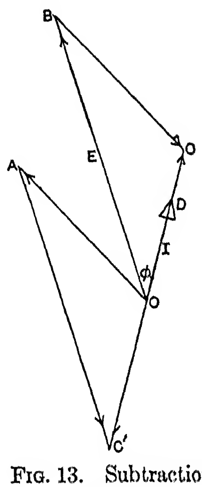

FIG. 13. Subtraction of vectors.

The preceding results with regard to the addition and subtraction of vectors are summed up in the following rule: *Relations which are true* **algebraically** *for instantaneous values of sinusoidal currents and voltages, hold true* **geometrically** *for the vectors of these quantities.* It is customary to provide vectors of currents with triangular arrows, as in Fig. 12; vectors of voltages are usually distinguished by pointed arrows, as in Fig. 13. This distinction enables one to see directly from the diagram whether a vector represents a current or a voltage, without reference to the text.

Prob. 1. The currents generated by two alternators in parallel are 75 and 120 amp. respectively, the second current lagging behind the first by 30 degrees. Determine the magnitude and the relative phase position

of the resultant line current by three methods: (a) point by point; (b) analytically; (c) by means of vectors.

Ans. 188.8 amp., lagging by 18° 32′ behind the first current.

Prob. 2. Solve the preceding problem without the use of eqs. (46) and (47), simply by means of the theorem proved above, that the sum or the difference of two sine-waves is also a sine-wave. Solution:

$$I_{eq} \sin (u + \phi_{eq}) = 75 \sin u + 120 \sin (u - 30°).$$

This equation is true for any instant, or for any value of u. It contains two unknown quantities, the amplitude and the phase position of the resultant curve. It is necessary, therefore, to apply this equation to two particular moments of time, in order to obtain two equations with two unknown quantities. It is most convenient in this particular case to choose $u = \pi/2$ and $u = 0$. Substituting these values, two equations with two unknown quantities are obtained. This method is preferable in the solution of practical problems, because it is not necessary to remember eqs. (46) and (47), and also because the two values of u can be selected so as to give the simplest equations.

Prob. 3. Two alternators, with the same number of poles, are coupled together so as to give voltages differing in phase by 27 degrees, the voltage of the second machine leading that of the first. The first alternator generates a voltage the amplitude of which is 2300 volts, the second 1800 volts. The two machines are connected electrically in series. Find graphically the vector of the resultant voltage in its magnitude and phase position. Find also the vector of the resultant voltage when the terminals of one of the machines are reversed. Ans. (1) 3988 volts, leading the first by 11° 49′; (2) 1074 volts, lagging behind the first by 49° 32′.

Prob. 4. An alternator, the terminal voltage of which is 6600, supplies its load through a transmission line. The conditions are such that the current lags behind the generator voltage by an angle of 35 degrees. The voltage drop in the line is 540 volts, leading the current in phase by an angle of 67 degrees. Find the receiver voltage by subtracting the voltage drop in the line from the generator voltage (geometrically); also determine the phase displacement between the receiver voltage and the current. Ans. 6149 volts; 32° 20′.

15. Non-sinusoidal Currents and Voltages. When a current or voltage wave differs considerably from the pure sine form, it is often convenient to represent it as the result of a superposition of sine-waves of different frequencies (Fig. 14). No matter how complicated a periodic wave may be, it can always be so represented with sufficient accuracy, by properly selecting the amplitudes and the phase relations of the component sine-waves, or *harmonics*, as they are called. Theoretically, an infinite number of sine-waves is necessary in order to represent any given irregular wave exactly. In practice, however, a limited number of harmonics is sufficient.

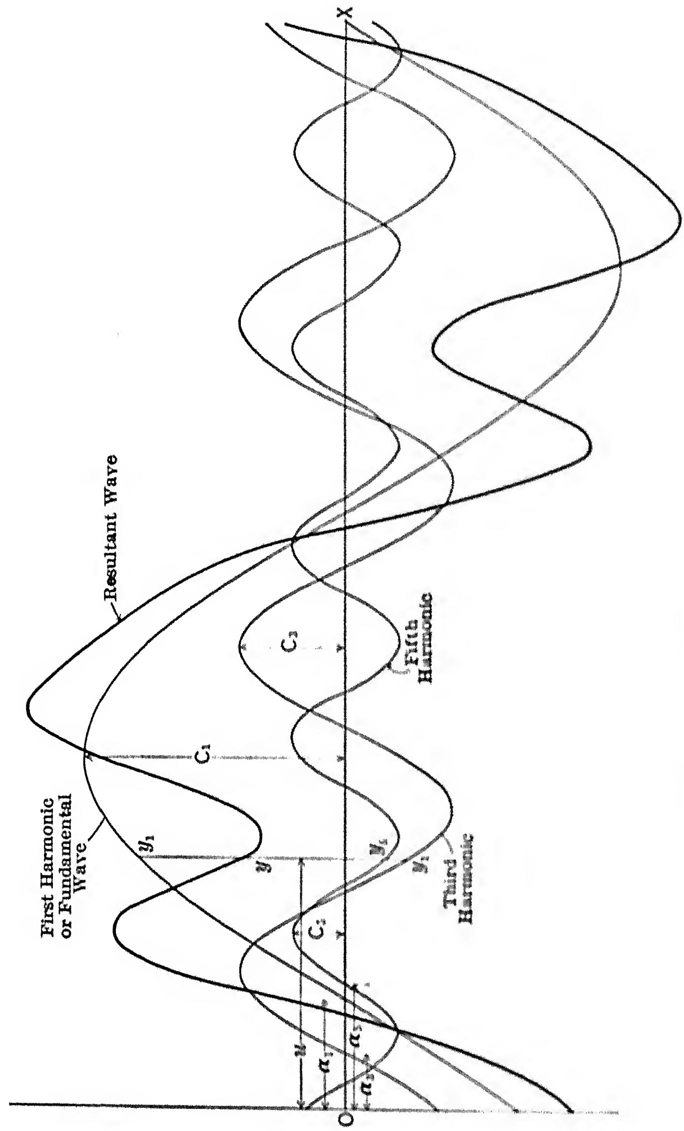

FIG 14. An irregular wave and its harmonics.

If the frequency of the given irregular wave is f, the frequencies of the harmonics are f, $2f$, $3f$, and so on. When, however, the given wave is symmetrical, that is, when the part above the axis of abscissæ is identical with that below, all *even* harmonics ($2f$, $4f$, etc.) drop out, and the wave consists only of the *fundamental* wave of frequency f, and the *odd* harmonics ($3f$, $5f$, etc.). The student can easily convince himself of the truth of this statement by taking a fundamental sine-wave and adding to it a second harmonic and a third harmonic. In the first case the resultant wave will be unsymmetrical; in the second, symmetrical. Nearly all of the waves encountered in practice are symmetrical.

Let the fundamental wave be represented by the equation $y_1 = C_1 \sin (u - \alpha_1)$, the third harmonic by the equation $y_3 = C_3 \sin 3(u - \alpha_3)$, etc. The meaning of C_1, C_3, etc., and of α_1, α_3, etc., is clear from Fig. 14; O is an arbitrary origin from which the angles are measured. The ordinates of the given composite symmetrical wave are represented by the equation

$$y = C_1 \sin (u - \alpha_1) + C_3 \sin 3(u - \alpha_3) + C_5 \sin 5(u - \alpha_5) + \text{etc.} \quad \ldots \ldots \ldots \ldots \quad (48)$$

This expression is known as the *Fourier series*, and is of great importance in mathematical physics.

In practice, the problem which presents itself is usually that of *analysis;* that is to say, it is often required to analyze or resolve a given irregular wave into its harmonics. In other words, knowing y, one is asked to determine the values of C and α for one or more harmonics. This is a purely mathematical problem, and is not treated here, because the solution will be found in numerous textbooks, handbooks and magazine articles.[1] There are also mechanical wave analyzers on the market, by means of which any desired harmonic may be separated by tracing the given curve with a stylus, in a manner similar to the way in which a planimeter is used.

It is of importance for an electrical engineer to train his eye in the discernment of prominent harmonics, without mathematical analysis. This training is afforded by exercises in wave *synthesis*, that is, in combining various assumed harmonics into irregular waves.

[1] See, for instance, the author's *Experimental Electrical Engineering*, Vol. 2, p. 222.

Take first a fundamental sine-wave and a third harmonic of a reasonable magnitude, say between 15 and 30 per cent of the fundamental. Combine these waves into one, with different relative phase positions of the fundamental and the third harmonic. In this way a flat wave, a peaked wave and a one-sided "humped" wave will be obtained. Then change the magnitude of the third harmonic and construct similar waves, in order to see the influence of this factor. After that, plot similar curves for the fundamental wave with a fifth harmonic, a seventh harmonic, and so on. Finally, combine the fundamental wave with the third and the fifth harmonics simultaneously, and so on. After some practice, the eye will easily discern prominent harmonics in a given irregular wave. Numerous oscillograms of irregular waves will be found in many current periodicals and in the transactions of the various electrical societies. Read in this connection Art. 30 of the *Magnetic Circuit*.

Prob. 1. Draw two or three sets of curves suggested in the preceding paragraph; each set must comprise about six curves and each curve must have a harmonic with a different phase position.

Prob. 2. Devise a simple apparatus by means of which harmonics can be combined mechanically, and the resultant waves observed, without actually plotting curves point by point.

Prob. 3. Analyze a given irregular wave into its harmonics, using the method given in the reference above, or any other method found in the literature on the subject.

CHAPTER V

POWER IN ALTERNATING-CURRENT CIRCUITS

16. Power when Current and Voltage are in Phase. Let a resistance r be connected across the terminals of an alternator, the voltage at the terminals varying according to the sine law. The current through the resistance also varies according to the sine law, because Ohm's law holds true for any moment of time, so that the curve of the current is in *phase* with that of the voltage. If the equation of the voltage wave is $e = E_m \sin u$, the equation of the current is $i = (E_m/r) \cdot \sin u$. Graphically, the current and the voltage are represented by two vectors of different lengths, but in the same direction — for instance, like OC and OD in Fig. 13.

Divide the time T of one cycle into a large number of small intervals Δt. Then the amount of energy delivered to the resistance r and converted into joulean heat during one of such intervals varies with the time position of the interval in the cycle, in other words, with the instantaneous values of the voltage and the current. This energy is practically equal to zero when the current and the voltage have values near zero, and it reaches a maximum with them. However, the dissipated energy, being in the nature of a frictional loss, never becomes negative, because whether the current flows in one direction, or in the other, the heat liberated, $e \cdot i \cdot \Delta t = i^2 r \cdot \Delta t$, is always positive.

Since the voltage and the current vary with the time, the rate of liberation of energy, or the instantaneous power, is also variable. The expression $P = ei = i^2 r$ represents the instantaneous power as with direct current. If e and i remained constant for one second, the energy liberated would be equal to $i^2 r$. As a matter of fact, e and i may be considered constant only during the infinitesimal element of time dt, so that the energy liberated during the time dt is $i^2 r \cdot dt$. Nevertheless, it is proper to say that at the instant under consideration the energy is liberated *at a rate* equal to $i^2 r$ per second, because $(i^2 r \cdot dt)/dt = i^2 r$. This is analogous to the

way in which we speak of the instantaneous speed of a body during a period of acceleration or retardation. The speed varies from instant to instant, so that to say that the speed is v at a certain instant merely means that, *if* the body continued to move at this velocity for one second, it *would* cover a space equal to v. In the same sense, the instantaneous power indicates the amount of energy which *would* be developed per second, *if* the current and the voltage suddenly became constant.

The total energy liberated in the form of heat during one complete cycle is

$$W = \int_0^T i^2 r \cdot dt = r \int_0^T i^2 \cdot dt. \quad . \quad . \quad . \quad . \quad . \quad (49)$$

When no local e.m.fs. are present, the same energy is represented by the expression

$$W = \int_0^T ei \cdot dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

When there are local e.m.fs. in the part of the circuit under consideration, the total energy communicated to it during an interval of time is different from that dissipated as heat (Art. 4). According to eq. (19), we have

$$W = \int_0^T e_t i \cdot dt + \int_0^T e_l i \cdot dt = r \int_0^T i^2 \cdot dt. \quad . \quad . \quad (51)$$

Suppose, for example, that e_l is the counter-e.m.f. of a motor in the circuit, and therefore nearly in phase opposition to e_t. Then the i^2r loss on the right-hand side of the equation is the difference between the energy supplied to the circuit and that converted into mechanical work in the motor.

The foregoing equations are true whether the current and the voltage vary according to the sine law or not. If they are sinusoidal, the integration can be easily performed, and the energy per cycle evaluated by the following method. Let the current be represented as before by $i = I_m \sin u$. Substituting this value into eq. (49), we have

$$W = I_m^2 r \int_0^T \sin^2 u \, dt. \quad . \quad . \quad . \quad . \quad . \quad (52)$$

This expression is easily integrated by using the substitution $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$. Or it may be evaluated by observing that its value remains the same if a cosine is substituted for the

sine. This is because the limits of integration are $u = 0$ and $u = 2\pi$, and in summing up sines or cosines through 2π we take the same quantities, only in a different order. Hence we may write

$$W = I_m^2 r \int_0^T \cos^2 u \, dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (52a)$$

Adding the two expressions term by term, and remembering that $\sin^2 u + \cos^2 u = 1$, we get

$$2W = I_m^2 r \int_0^T dt = I_m^2 r \cdot T,$$

or the energy converted into heat during one cycle is

$$W = \tfrac{1}{2} I_m^2 r \cdot T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

When there are no local e.m.fs. and the current is in phase with the voltage, we have $E_m = I_m r$, so that from eq. (53), and by analogy to eq. (18), we have

$$W = \tfrac{1}{2} I_m E_m \cdot T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

and

$$W = \tfrac{1}{2} (E_m^2 / r) \cdot T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

The student must clearly understand that the phase relation between the current and the voltage is of no consequence in eq. (53), while eqs. (54) and (55) hold true only when the current is in phase with the voltage. Or else, E_m in these latter expressions may be said to refer to that component of the total terminal voltage which is used up in Ir drop.

Prob. 1. A sine-wave alternating current, which fluctuates between ± 75 amp., flows through a resistance of 10 ohms. Plot curves of instantaneous values of the voltage and power; the frequency is 50 cy./sec.

Ans. $E_m = 750$ volts; max. power $= 56.25$ kw.

Prob. 2. Determine the total energy liberated per cycle in the preceding problem, by integrating graphically the curve of power.

Ans. 562.5 joules (watt-seconds).

Prob. 3. Prove analytically that the curve of power obtained in problem 1 is a sine-wave of double frequency, tangent to the axis of time. Proof: The equation of the curve is $P = I_m^2 r \cdot \sin^2 u$. But from trigonometry

$$\cos 2u = \cos^2 u - \sin^2 u = 1 - 2 \sin^2 u.$$

Substituting the value of $\sin^2 u$ from this equation into the expression for P, we get

$$P = \tfrac{1}{2} I_m^2 r - \tfrac{1}{2} I_m^2 r \cdot \cos 2u.$$

The first term is constant, while the second represents a sine-wave of

double frequency, because $2u = 2\pi(2f)t$. The first term is never smaller than the second, so that P is always positive, and the whole curve lies above the axis of abscissæ. The second term becomes equal to the first and $P = 0$, only when $2u$ is a multiple of 2π. At these points the curve is tangent to the axis of abscissæ.

Prob. 4. Deduce eq. (53) directly from (52), expressing $\sin u$ in terms of the cosine of the double angle, as in the preceding problem. Hint: From eq. (37), $dt = (T/2\pi)\,du$, and the limits of integration are $u = 0$ and $u = 2\pi$.

17. The Effective Values of Current and Voltage. In practice, it is *the average rate* of delivery or dissipation of energy that is of interest, or, in other words, the average value of the variable instantaneous power. This is analogous to using in calculations the average speed of a machine, when the actual speed varies within certain limits. This average power is found by dividing the total energy developed during one cycle by the period T of the cycle. When the current varies according to the sine law, the total energy per cycle converted into heat is expressed by eq. (53). Dividing both sides by T, we find that the average power

$$P_{ave} = \tfrac{1}{2} I_m^2 r. \qquad (56)$$

It is convenient to use in eq. (56) a new value of the current,

$$I = I_m/\sqrt{2} = 0.707\, I_m, \qquad (57)$$

instead of I_m, because then the expression for the average power becomes identical with that in a direct-current circuit, namely,

$$P_{ave} = I^2 r. \qquad (58)$$

Analogously, if we define

$$E = E_m/\sqrt{2} = 0.707\, E_m, \qquad (59)$$

eqs. (54) and (55) become

$$P_{ave} = E \cdot I \qquad (60)$$

and

$$P_{ave} = E^2/r, \qquad (61)$$

which are perfectly similar to the corresponding expressions in a direct-current circuit.

E and I, as defined above, are called the *effective* values of the alternating voltage and current respectively. We may say that by definition the effective value of an alternating (or variable) current is equal to such a constant current which, when flowing

through a resistance, dissipates the same *average* power as the actual variable current.

This definition of an effective value applies to variable currents of any form. It is used, for instance, in determining the temperature rise of electric railway motors. During the run of a car the current fluctuates within wide limits, but the heating of the motor windings is nearly the same as would occur with a certain constant current, which is called the effective value of the actual variable current. The condition for the same average i^2r loss is

$$I^2r \cdot T = r\int_0^T i^2\,dt,$$

where T is the interval of time for which it is desired to obtain the effective value. Hence

$$I^2 = (1/T)\int_0^T i^2\,dt. \quad \ldots \ldots \quad (62)$$

This equation expresses in mathematical language that I^2 *is the average value of* i^2, over the period of time T. Taking the square root of both sides of this equation, we can also define the effective value I as *the square root of the mean square of the instantaneous values*. This definition is true for any form of alternating or variable current. The effective voltage is defined by a similar expression, so that more generally

$$y_{eff}^2 = (1/T)\int_0^T y^2\,dt, \quad \ldots \ldots \quad (63)$$

where y denotes an instantaneous value of current or voltage. Alternating-current ammeters and voltmeters are always calibrated so as to indicate the effective values of current and voltage.

When an irregular wave of current or voltage is given graphically, its effective value is found by taking a sufficient number of equidistant ordinates (Fig. 15) and replacing the integration in eq. (63) by a summation. Let the half-wave be divided into k equal parts, where k is an even number, and let $y_0, y_1, \ldots y_k$ be the corresponding ordinates. Then, according to Simpson's Rule,

$$y_{eff}^2 = [1/(3k)]\,[(y_0^2 + y_k^2) + 4(y_1^2 + y_3^2 + \cdots + y_{k-1}^2) + 2(y_2^2 + y_4^2 + \cdots + y_{k-2}^2)] \quad . \; . \quad (63a)$$

The larger the number of ordinates, the more accurate is the value of y_{eff} determined by this method. The value of y_{eff}^2 may also be found by plotting a curve of y^2 against u, as shown in Fig. 15, and determining its mean ordinate by means of a planimeter.

When a large number of effective values must be determined, —for instance, from the records obtained by a graphic ammeter during several runs of an electric train, the squaring of ordinates becomes a tedious process. Some practical methods, by means of which the necessity for squaring ordinates is eliminated, are described in the next article.

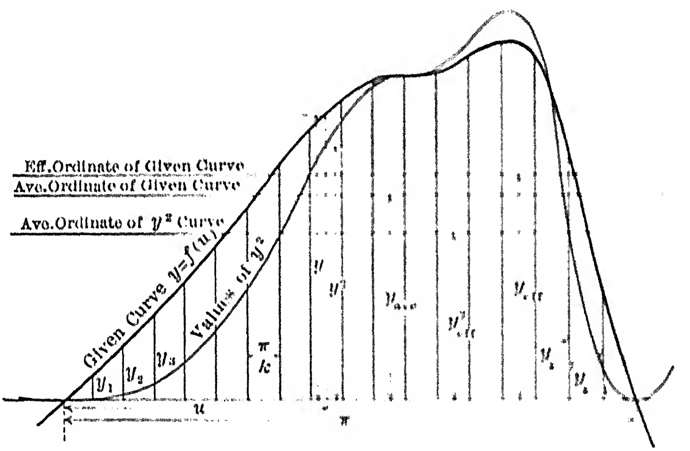

FIG. 15. The effective and the average ordinates of an irregular half-wave.

The effective value of a current or a voltage is also called the *quadratic mean*, to distinguish it from the arithmetical mean value defined by the familiar equation

$$y_{ave} = (1/T)\int_0^T y\,dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

or for a periodic curve

$$y_{ave} = (1/\pi)\int_0^\pi y\,du. \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

It will be noted that the upper limit of integration is π and not 2π. It is evident that for a symmetrical wave the average ordinate over a whole cycle is equal to zero. The average value, therefore, always refers to a half-wave.

For the sine-wave

$$y_{ave} = (y_m/\pi)\int_0^{\pi} \sin u \, du = (2/\pi)y_m,$$

or

$$y_{ave}/y_m = 2/\pi = 0.637. \quad . \quad . \quad . \quad . \quad . \quad (66)$$

The ratio of the effective to the mean ordinate is called the *form factor*, because it gives an idea of the degree to which the curve is flat or peaked as compared to the sine-wave. For a sine-wave the form factor is

$$(y_m/\sqrt{2})/(2\,y_m/\pi) = 1.11. \quad . \quad . \quad . \quad . \quad . \quad (67)$$

For a perfectly flat-topped or rectangular wave, the maximum value, the effective value and the average value are all the same, so that the form factor is equal to unity. For very peaked waves, the influence of the high middle ordinates is more prominent in the quadratic mean, so that the effective is considerably higher than the mean value, and the form factor is larger than 1.11.

Another ratio which helps in judging about the shape of a curve is the so-called *amplitude factor*, or the ratio of the maximum ordinate to the effective value. The author is not aware that either the form factor or the amplitude factor is used to any considerable extent in practice.

Prob. 1. An electric heater was tested for power consumption on an alternating-current circuit, by having an ammeter in series with it, and a voltmeter across its terminals. Both instruments were calibrated to indicate effective values. The readings were 110 volts and 5.7 amp. Assuming the current and the voltage to have been in phase, which is nearly the case, what was the average power consumption of the heater, and what was its resistance? Determine also the maximum instantaneous values of the current and the voltage, under the supposition of the sine law. Ans. 627 watts; 19.3 ohms; 155.56 volts; 8.06 amp.

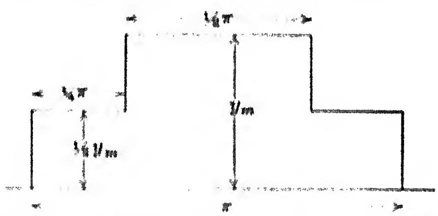

FIG. 16. A stepped curve of current or voltage.

Prob. 2. Determine the average value, the effective value, the form factor and the amplitude factor of the curve shown in Fig. 16.

Ans. 0.75 y_m; 0.791 y_m; 1.055; 1.264.

Prob. 3. Check some of the values of the form factor and the amplitude factor given in the table in the *Standard Handbook* (see Index under "form factor"). This will afford practice in calculating effective values of curves when they are given by analytic equations of the form $y = f(t)$, using eq. (63).

Prob. 4. Plot an irregular wave, taken from an available oscillograph record, and calculate its average and effective values by the point-by-point method, or by using a planimeter.

18. Some Special Methods for Calculating the Effective Value of an Irregular Curve. As is mentioned in the preceding article, squaring a large number of ordinates in order to find the effective value of a curve is a tedious process, and methods are available which sometimes lead to the end more quickly. It must be admitted, however, that for one who has to do this work only occasionally, the plain point-by-point method described above is probably the quickest and the most reliable.

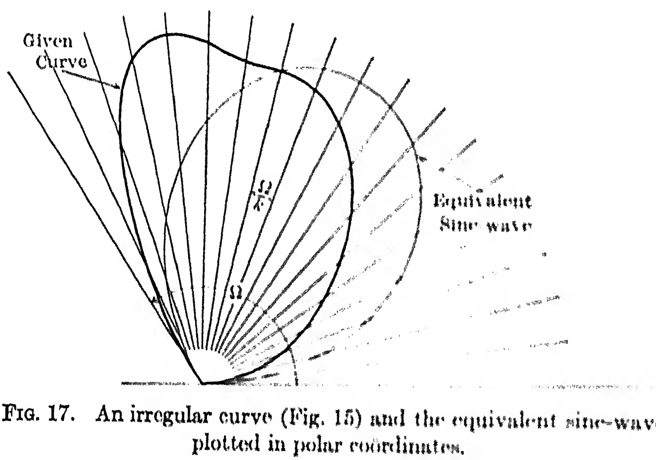

FIG. 17. An irregular curve (Fig. 15) and the equivalent sine-wave, plotted in polar coördinates.

(a) *Fleming's Method.* The given curve (Fig. 15) is replotted in polar coördinates (Fig. 17), so that equal polar angles Ω/k correspond to equal distances π/k upon the axis of abscissæ. The ratio between an abscissa u in Fig. 15 and the corresponding polar angle in Fig. 17 is of no consequence; in other words, it makes no difference what total central angle Ω corresponds to the total distance π in Fig. 15. The area of an infinitesimal triangle subtended by a polar angle $d\omega$ is $\frac{1}{2}\, y^2 d\omega$, because y is the base of the

triangle, and $y\,d\omega$ is its altitude. Thus, the total area of the curve in Fig. 17 is

$$S = \tfrac{1}{2}\int_0^{\Omega} y^2\,d\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

But, by the defining equation (63), the effective value, or the quadratic mean ordinate, of the same curve in Fig. 15 is found from the expression

$$y_{eff}^2 = (1/\Omega)\int_0^{\Omega} y^2 d\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

because ω is proportional to t. Comparing the preceding two equations, we find that

$$y_{eff}^2 = 2\,S/\Omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Since S is easily evaluated, for instance, by means of a planimeter, the effective value is calculated from eq. (70) without squaring the ordinates, but simply by replotting the given curve in polar coördinates.[1]

When the given curve is a pure sine-wave, the corresponding curve in polar coördinates is a circle, provided that the angle Ω is selected equal to π. The student can easily prove this for himself, either graphically or analytically. Let y_m be the maximum ordinate of the sine-curve; then the area of the circle is $S = \frac{1}{4}\,\pi y_m^2$, and from eq. (70) we find $y_{eff} = y_m/\sqrt{2}$. This is the same value as found before by a different method.

When the given curve is not much different from a pure sine-wave, the corresponding polar curve approaches a circle in form (always provided that $\Omega = \pi$). In such cases it is possible to determine the area of the polar curve without a planimeter, by drawing a circle of equal area as judged by the eye (Fig. 17). The effective value is then the same for the given curve and for the sine-wave corresponding to this circle, and is equal to the diameter of the circle divided by $\sqrt{2}$. Such a sine-wave is called the *equivalent* sine-wave. It is often convenient in dealing with irregular current and voltage waves to replace them by equivalent sine-waves, so as to be able to apply an analytical solution, or to construct vectors.

[1] For a more detailed treatment and numerous practical applications, see C. O. Mailloux, "Méthode de Détermination du Courant Constant Produisant le même Echauffement qu'un Courant Variable," in the *Transactions of the International Congress of Applications of Electricity*, Turin, 1911.

(b) *The Effective Value in Terms of Harmonics.* When an irregular wave is given in the form of a Fourier series, eq. (48), the effective value can be expressed through the amplitudes of the harmonics. In order to use the expression for y in the fundamental formula (63), we have to square the Fourier expansion. This gives terms of two kinds, namely, squares of harmonics, and products of pairs of harmonics. Let the nth and the pth harmonics be represented by the expressions

$$h_n = C_n \sin n(u - \alpha_n) \; . \; . \; . \; . \; . \; . \; . \quad (71)$$

and

$$h_p = C_p \sin p(u - \alpha_p) \; . \; . \; . \; . \; . \; . \; . \quad (72)$$

Then the right-hand side of eq. (63) will contain the following terms:

$$(1/T)\int_0^T h_n^2 dt = (C_n^2/T)\int_0^T \sin^2 n(u - \alpha_n) dt = \tfrac{1}{2} C_n^2; \quad (73)$$

$$(1/T)\int_0^T h_p^2 dt = (C_p^2/T)\int_0^T \sin^2 p(u - \alpha_p)\, dt = \tfrac{1}{2} C_p^2; \quad (74)$$

$$2\,(1/T)\int_0^T h_n h_p dt =$$

$$2\,(C_n C_p/T)\int_0^T \sin n(u - \alpha_n) \sin p(u - \alpha_p)\, dt = 0. \; . \; . \; . \; . \quad (75)$$

The values of the first two integrals are found in precisely the same way as that of eq. (52) in Art. 16, that is, on the basis of the fact that their values do not change if cosines are substituted for the sines. The third integral is identically equal to zero, as is shown in problem 3 below. Thus eq. (63) becomes

$$y_{eff}^2 = (C_1/\sqrt{2})^2 + (C_3/\sqrt{2})^2 + \text{etc.}, \; . \; . \; . \quad (76)$$

or the square of the effective value of a complex wave is equal to the sum of the squares of the effective values of its harmonics.

Prob. 1. Plot a complex wave consisting of known harmonics and determine its effective value (a) by the method given in the preceding article; (b) by the Fleming method; (c) from eq. (76).

Prob. 2. An irregular wave has a third and a fifth harmonic, the amplitudes of which are equal respectively to 12 per cent and 4 per cent of that of the first harmonic. Show that the effective ordinate is equal to 71.3 per cent of the amplitude of the fundamental wave, and that the average value depends upon the phase positions of the harmonics.

Prob. 3. Prove that expression (75) is identically equal to zero. Proof: According to the familiar formula of trigonometry, $\sin A \sin B = \frac{1}{2}\cos(A - B) - \frac{1}{2}\cos(A + B)$, we have $\sin n(u - \alpha_n) \sin p(u - \alpha_p) = \frac{1}{2}\cos[(n - p)u + a] - \frac{1}{2}\cos[(n + p)u + b]$, where a and b do not contain the variable u. Integrating these cosines leads to terms of the form $\sin[(n - p)u + a]$ and $\sin[(n + p)u + b]$. Since the limits of integration are 0 and 2π, and n and p are integers, the values of these sines at the upper limit are the same as at the lower limit, and consequently each of the integrals is equal to zero.

19. Power when Current and Voltage are out of Phase. In a majority of practical alternating-current circuits there is a more or less pronounced phase displacement between the current and the voltage. This is due to the presence of local electromotive forces, the principal among these being as follows: (a) The counter-electromotive forces of motors connected into the circuit. (b) The electromotive forces induced by alternating magnetic fluxes in the circuit. These fluxes may be created by the current itself, or they may be due to the influence of other circuits (self and mutual induction). (c) The electromotive forces due to the "elastivity" of the dielectric medium surrounding the circuit (electrostatic capacity or permittance).

The actual workings of these causes are discussed more in detail in the following chapters. Here it is sufficient to note that there are factors which produce local electromotive forces in alternating-current circuits, and that they bring about a phase displacement between the voltage and the current. Let OB (Fig. 13) be the generator voltage, and let OA represent the sum of the various local electromotive forces in the circuit. Subtracting OA from OB, the net voltage OC is obtained, which is just sufficient to supply the ohmic drop in the circuit. The current OD is in phase with this voltage, and is numerically equal to OC divided by the total resistance r of the circuit. It will be seen that there is a phase displacement ϕ between the current and the generator voltage OB; it is also clear from the figure that this phase displacement is due to the presence of the electromotive force BC.

We shall first calculate the energy supplied by the generator during one cycle in the specific case when the phase displacement between the current and the voltage is exactly 90 degrees. If the current is represented by the equation $i = I_m \sin u$, the expression for the voltage is $e = E_m \cos u$. The instantaneous

power is equal to $i \cdot e = I_m E_m \sin u \cos u = \frac{1}{2} I_m E_m \sin 2u$. Thus, the power varies as a sine function of double the generator frequency; the energy flows now away from, and now toward, the generator. The *average* power for one cycle is therefore zero, for the power has as many negative values as it has positive ones. Mathematically, this result is represented by the time integral of the instantaneous power over a complete cycle. Omitting the constant quantities E_m and I_m, we have

$$\int_0^{2\pi} \sin u \cos u \, du = \tfrac{1}{4}\Big[-\cos 2u\Big]_0^{2\pi} = 0.$$

Let now the phase displacement between the current and the voltage be less than 90 degrees, and be equal, say, to ϕ. The average power delivered by the alternator is in this case smaller than the product EI, and its value must be investigated. The vector of the voltage E can be resolved into a component $E \cos \phi$ in phase with the current, and another component, $E \sin \phi$, in quadrature with the current. According to the proof given above, the average power produced by the quadrature component of the voltage is zero, so that the total average power is

$$P_{ave} = EI \cdot \cos \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

A more rigid proof of this expression is given in problem 3 below.

The product EI is called *the apparent power*, and $\cos \phi$ is referred to as *the power-factor*. Thus, the power-factor can be defined either as the cosine of the angle of phase displacement between the current and the voltage, or as the ratio of the true power P_{ave} to the apparent power IE. The second definition is more general, because it applies also to non-sinusoidal currents and voltages.

Referring to Fig. 13, the factor $I \cos \phi$ which enters into eq. (77) represents the projection of I upon the direction of the voltage OB, or E. Hence, eq. (77) can be interpreted by saying that the true power is equal to the product of the voltage by the component of the current in phase with it. This component of the current, $I \cos \phi$, is therefore called *the energy component*, while the component $I \sin \phi$, at right angles or in quadrature with the voltage, is called *the reactive component*.[1]

Instead of resolving the vector of the current into two components, it is sometimes preferable to resolve the voltage E into

[1] The older name for this reactive component is wattless current.

the components $E \cos \phi$ and $E \sin \phi$, in phase and in quadrature with the current. In this case, eq. (77) is expressed in words by saying that the average power is equal to the current times the component of the voltage in phase with it. These components of the voltage are also called *the energy component* and *the reactive component* respectively. The two components of power, the true power $EI \cos \phi$, and the reactive power $EI \sin \phi$, stand in the same relation to the apparent power EI as the two sides of a right triangle bear to the hypotenuse; that is,

$$(EI)^2 = (EI \cos \phi)^2 + (EI \sin \phi)^2. \quad . \quad . \quad . \quad . \quad (78)$$

Let now the current and voltage curves be different from pure sine-waves, and also different from each other in form. The fundamental equation

$$P_{ave} = (1/T) \int_0^T ei \cdot dt \quad . \quad . \quad . \quad . \quad . \quad (79)$$

holds true in all cases, so that if the curves are given graphically, the energy per cycle is found by multiplying the corresponding instantaneous values of e and i, and using the planimeter on the resultant curve. The average ordinate of this curve gives the average power. Of course, the parts of the resultant curve below the axis of abscissæ must be evaluated separately from those above it, and the difference of the two taken to represent the total energy.

If the two waves are given in the form of Fourier series, an expression for the average power may be obtained in terms of the effective values of the harmonics. Substituting the expansions for e and i into eq. (79), two kinds of terms are obtained,—those containing products of two harmonics of the same frequency, and those containing products of two harmonics of different frequencies. The terms of the first kind, after integration, give results of the same form as for the fundamental wave; that is, for the nth harmonic $\frac{1}{2} E_n I_n \cos \phi_n$, where E_n and I_n are the amplitudes of the nth harmonics, and ϕ_n is the phase displacement between them. The terms of the second kind give zero after integration, the proof of this being analogous to that in problem 3 of the preceding article. Thus

$$P_{ave} = \tfrac{1}{2} E_1 I_1 \cos \phi_1 + \tfrac{1}{2} E_3 I_3 \cos \phi_3 + \text{etc.} \quad . \quad . \quad . \quad (80)$$

In other words, *each harmonic contributes its own share of power, as if it were acting alone.*

Let E and I be the effective values of some non-sinusoidal periodic voltage and current, measured, for instance, by means of hot-wire or dynamometer-type instruments. Let P_{ave} be the average power according to eq. (80), or measured by a dynamometer-type wattmeter. Then the ratio P_{ave}/EI is called the power-factor of the system, the same as with sinusoidal curves. This ratio is also often denoted by $\cos\phi$, meaning by ϕ the phase angle between the *equivalent* sine-waves of voltage and current, as defined in the preceding article. With the use of this angle and of the equivalent sine-waves, vector diagrams may be constructed and the corresponding calculations performed with currents and voltages deviating considerably from pure sine-waves, though of course such calculations check only approximately with the actual measurements.

Prob. 1. Assuming the line current in problem 4, Art. 14, to be 452 effective amperes, calculate the average power delivered by the alternator, and the power received at the opposite end of the line.

Ans. 2444 kw.; 2350 kw.

Prob. 2. Referring to problem 1, Art. 17, a wattmeter was connected into the heater circuit, and the true power was found to be 598 watts. Assuming all the three instruments to be in calibration, calculate the power-factor and the angle of displacement between the current and the voltage in the heater; also the energy component and the reactive component of the current.

Ans. 95.1 per cent; 17° 30'; 5.39 amp.; 1.71 amp.

Prob. 3. Deduce expression (77) for power by direct integration. Solution: Let the current be expressed by $I_m \sin u$; also let the voltage be leading by an angle ϕ, and therefore expressed as $E_m \sin(u + \phi)$. Substituting these values into eq. (79), we get

$$P_{ave} = (E_m I_m/T)\int_0^T \sin u \sin(u + \phi)\,dt,$$

$$= (E_m I_m/2\pi)\int_0^{2\pi} \sin u\,[\sin u \cos\phi + \cos u \sin\phi]\,du,$$

$$= (E_m I_m/2\pi)\,[\cos\phi\int_0^{2\pi}\sin^2 u\,du + \sin\phi\int_0^{2\pi}\sin u \cos u\,du].$$

From a table of integrals we find that the value of the first integral is π, and that of the second is zero. Substituting these values, and introducing the effective values of voltage and current, formula (77) is obtained.

Prob. 4. Plot a sine-wave representing an alternating voltage of 500 effective volts, and a current of the same frequency, of 20 effective amperes, lagging behind the voltage by 30 degrees. Plot on the same curve sheet the sine-wave of the instantaneous power, and check the

average ordinate of this curve with the value obtained by formula (77). Explain why the power is negative during a part of the cycle, remembering that there are local electromotive forces in the circuit.

Prob. 5. Prove that the curve of power consists of a sine-wave of double frequency, plus a constant term, the latter representing the average power. (Compare with problem 3, Art. 16. Suggestion: $ie = I_m E_m \sin u \sin (u + \phi)$. Use the trigonometric transformation, $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$.

Prob. 6. A non-sinusoidal voltage is represented by the equation $e = 270 \sin u + 62 \sin 3 (u + 15^\circ) + 16 \sin 5 (u - 25^\circ)$; the corresponding line current is $i = 18 \sin (u - 30^\circ) - 7 \sin 3 (u + 50^\circ) + 2.5 \sin 5 (u + 10^\circ)$. Calculate the true average power and the power-factor of the system.

Ans. $P_{ave} = \frac{1}{2} (4860 \cos 30^\circ + 434 \cos 75^\circ - 40 \cos 5^\circ) = 2141$ watts; the power-factor is 80.3 per cent.

CHAPTER VI

INDUCTANCE, REACTANCE AND IMPEDANCE

20. Inductance as Electromagnetic Inertia. Experiment shows that an electric current in a variable state behaves as if it possessed inertia; there is an opposition to any change in its magnitude and direction. This opposition is manifested in the form of an "induced" electromotive force in such a direction as to tend to counteract the change in current. Thus, if an external e.m.f. tends to increase the current, the induced e.m.f. is in a direction opposite to that of the current; but when, on the other hand, the current for some reason decreases, the induced e.m.f. is in the same direction as the current, and therefore tends to strengthen it. These reactions of the current are similar to those exerted by a moving body; for instance, the water in a pipe, when its motion is accelerated or retarded. In practical applications, it is convenient to consider, not the reactions themselves, but the external forces necessary to overcome them. Thus, in the case of a moving body of mass m, the external force necessary to communicate to it an acceleration dv/dt is $F = m\,dv/dt$. Here F is positive when the acceleration is positive, and vice versa. Similarly, to increase a current at a rate of di/dt, an external e.m.f. is necessary of the magnitude

$$e = L\,di/dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

where L is a constant which characterizes the circuit and is analogous to the mass m in the mechanical motion. The coefficient L is called the *inductance* of the circuit, and depends upon its shape and proportions, the presence or absence of iron, the number of turns which the conductor makes, and some other factors, which it is not necessary to discuss here. In most books the right-hand side of eq. (81) is written with the sign minus, because e is understood to mean the induced e.m.f. or the reaction of the circuit; while in our case e designates the external voltage, equal and

opposite to this reaction. The form of the equation used here is preferable, because in practice one deals with components of the applied voltage rather than with the induced counter-e.m.f.; moreover, the minus sign is apt to confuse a beginner.

The inertia effect of the electric current is brought about through the mechanism of the magnetic field produced thereby. When the current varies, the flux embraced by the electric circuit also changes, and according to Faraday's law of induction this flux induces in the circuit an e.m.f. Thus, postulating the existence of electromagnetic inertia, and stating the law of induced e.m.f., are perhaps but two different ways of expressing the same physical phenomenon, the true nature of which is at present unknown. Any arrangement of the circuit which increases the flux linked with it, also increases its inductance L or the inertia effect. The inductance of a given electric circuit can be calculated with more or less accuracy,[1] or it can be measured experimentally, using eq. (81). For our present purposes we shall assume L to be a constant quantity, which characterizes the inertia of a given electric circuit, according to eq. (81), without any reference to the nature of the magnetic flux which produces it. Mechanical inertia is used in physics and in engineering as a fundamental entity, without explaining it in any other terms, while the mystery as to its cause is just as deep as that surrounding the electromagnetic inertia. Some modern physicists even believe that all inertia is of an electromagnetic nature.

The fact that a body resists acceleration, together with the law of conservation of energy, leads to the conclusion that a moving body possesses a certain amount of stored energy. The external work done upon a body while it moves through a distance ds is $F \cdot ds = m(dv/dt)ds$, or, since $ds = v \cdot dt$, we have $F \cdot ds = mv \cdot dv$. The total work done upon the body while accelerating it from rest to a velocity v is therefore

$$W = \int_0^s F\,ds = \int_0^v mv \cdot dv = \tfrac{1}{2}\,mv^2.$$

According to the law of conservation of energy, this work is stored in the moving body as its kinetic energy.

The electrical work done in increasing a current against the induced electromotive force, during the time dt is $dW = ei \cdot dt$, or

[1] See the author's *Magnetic Circuit*, Chapters 10 to 12.

substituting for e its value from eq. (81), $dW = Li\,di$. The total energy supplied to the circuit from the external source of power, while the current increases from zero to a certain value i, is

$$W = \int_0^i Li\,di = \tfrac{1}{2}Li^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

This does not include the energy required for supplying the i^2r loss. According to the law of conservation of energy, expression (82) represents the energy stored in the circuit as long as the value of the current remains the same. When the circuit is broken, this energy is converted into heat. Analogously, when a non-elastic moving body is stopped, its accumulated energy is converted into the heat of impact. Inductance can be defined from either eq. (81) or (82); and for most purposes the two definitions are identical. Similarly, in mechanics, mass may be defined either as the ratio of F to dv/dt, or as a ratio of the kinetic energy to $\frac{1}{2}v^2$.

The unit of inductance in the ampere-ohm system is called the *henry*. According to eq. (81), a circuit has an inductance of one henry when one volt is necessary in order to increase the current at a rate of one ampere per second. This one volt does not include, of course, the e.m.f. necessary for overcoming the resistance of the circuit. The henry being rather a large unit, inductance is frequently measured in millihenrys. Substituting into eq. (81) the physical dimensions of the voltage in the ampere-ohm system, we get [IR] = [LI/T] or [L] = [RT]. In other words, the henry stands for the "ohm-second." For this reason, one instrument for measuring inductance has been called by its inventors "the secohmmeter."

All actual circuits which possess inductance, at the same time, have some resistance, however small it may be. Therefore, the total instantaneous voltage applied during a variable state is

$$e = ir + L\,di/dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Ohmic resistance may be compared to mechanical friction, so that eq. (83) can be interpreted by reference to the mechanical analogy used above, in the following way; namely, the force necessary to accelerate a body must be augmented in practice by the amount required for overcoming the inevitable friction.

Prob. 1. A circuit which possesses an inductance of 12 millihenrys and carries a direct current of 150 amp. is broken within one-fifth of a second. What is the average voltage induced in the circuit during this interval of time?
Ans. 9 volts.

Prob. 2. Calculate the electromagnetic energy stored in the circuit of the preceding problem while the current is steady. Ans. 135 joules.

Prob. 3. The current in a coil is made to vary at a uniform rate of 250 amp. per second. At the instant when the current is equal to 150 amp., a voltmeter connected across the terminals of the coil reads 295 volts; when the instantaneous current is 100 amp. the voltmeter reading is 230 volts. From these data calculate the resistance and the inductance of the coil.
Ans. 1.3 ohms; 0.4 henry.

21. Reactance. It is natural to expect the inductance to exert a considerable influence upon the voltage and current relations in an alternating-current circuit, because the current is varying in magnitude all the time. The influence of inductance in this case is analogous to that of the inertia of the moving parts in a reciprocating engine; *i.e.*, energy is stored during the periods of increase in velocity (or in current), and is returned to the source of power during the intervals of time when the velocity (or the current) decreases. There is no net gain or loss of energy for a complete cycle, although the instantaneous values of current and voltage may be considerably affected.

Consider first a part of a circuit which has inductance only, the resistance being negligible. Let the current vary according to the familiar law $i = I_m \sin(2\pi ft - \alpha)$. Substituting this value into eq. (81), we get

$$e = 2\pi fLI_m \cos(2\pi ft - \alpha), \quad . \quad . \quad . \quad . \quad (84)$$

which means that the voltage necessary to force a sinusoidal current through an inductance also varies according to the sine law, and is in leading quadrature with the current. The amplitude of the voltage $E_m = 2\pi fLI_m$, or the relation between the effective values of voltage and current, is

$$E = 2\pi fLI. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

It will be seen from this relation that, in alternating-current calculations, the quantities f and L always appear as a product. It is therefore convenient to introduce, for the sake of abbreviation, a new composite quantity x, defined by the relation

$$x = 2\pi fL. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

The quantity x is called the *reactance* of the circuit, and always refers to a stated frequency f. Equation (85) becomes then

$$E = xI, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

from which it follows that reactance is measured in ohms, like resistance. This does not mean, however, that the two quantities are similar in their physical nature.

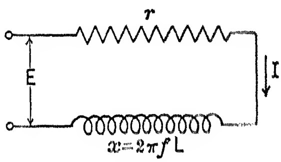

FIG. 18. Resistance and reactance in series.

Let now some resistance be connected in series with the inductance (Fig. 18), or let the coil which possesses inductance have also an appreciable resistance. Substituting the expression for i, given above, into eq. (83), we get

$$e = rI_m \sin (2\pi ft - \alpha) + xI_m \cos (2\pi ft - \alpha). \quad . \quad . \quad (88)$$

Since the sum of two sine-waves is also a sine-wave (see Art. 14), the total voltage e varies according to the sine-law. These component sine-waves of voltage, their sum, and the current wave are shown in Fig. 19. The student is advised to study this figure

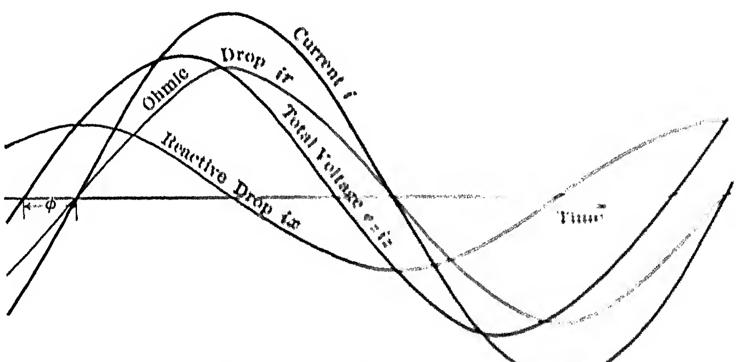

FIG. 19. The instantaneous current and voltage relations in the circuit shown in Fig. 18.

very carefully, because it represents one of the most important fundamental relations in the whole theory of alternating currents. The same relations are represented vectorially in Fig. 20, and the two figures may be conveniently examined together. One describes the phenomenon from instant to instant; the other gives

the salient features in a symbolic form. The vector E consists of one component Ir in phase with the current, and another Ix in leading quadrature with the current. The first component

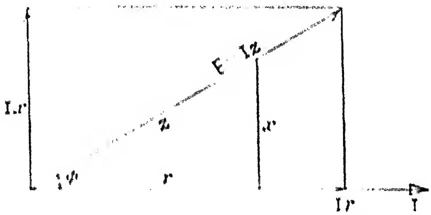

FIG. 20. The current and voltage relations in the circuit shown in Fig. 18, represented vectorially.

serves to overcome the ohmic resistance; the second, the reactance of the circuit. From the triangle of voltages we have

$$E^2 = I^2r^2 + I^2x^2,$$

or

$$E = I\sqrt{r^2 + x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

For the phase displacement between the current and the voltage we have

$$\tan\phi = Ix/Ir = x/r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

or the power factor

$$\cos\phi = r/\sqrt{r^2 + x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

The hydraulic analogue shown in Fig. 21 may make these relations clearer. $ACDGA$ represents a closed pipe circuit in which water is made to oscillate to and fro by means of the piston B. The water is assumed to be devoid of inertia, and the inertia of the whole circuit is concentrated in a heavy mass F, which moves freely with the water. The force upon the piston rod H is analogous to the alternating voltage E in Fig. 18; the velocity of the water is analogous to the alternating current, the friction in the pipes represents the ohmic resistance r, and the inertia of the heavy mass F stands for the inductance L. To make the analogy closer, we assume that the piston is forced to perform a simple harmonic motion;

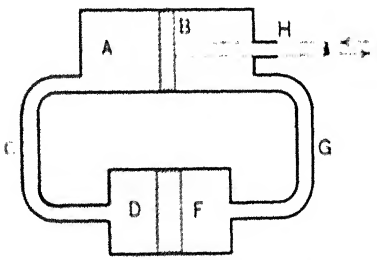

FIG. 21. A hydraulic analogue to Fig. 18.

so that the velocity of the water varies with the time according to the sine law, and may be represented by the curve for i in Fig. 19.

The force upon the piston B consists of two parts, that required for overcoming the friction in the pipes, and that necessary for accelerating and retarding the mass F. These two components of the force can be represented by the curves for ir and ix in Fig. 19. The frictional reaction is at a maximum when the piston is in the middle of its stroke, because there the velocity of the water is the greatest. On the other hand, the acceleration is zero in this position, so that the mass F exerts no reaction. At the ends of the stroke the acceleration or retardation is at a maximum, so that the force necessary for constraining the mass F to the prescribed motion is at a maximum; however, the frictional resistance is equal to zero. Adding the two sinusoidal components, we find the resultant force upon B, corresponding to the curve e in Fig. 19. It will be seen that e reaches a maximum *before* the center of the stroke; this gives a phase angle between the force and the velocity that is analogous to the phase angle between the voltage and the current. The student can easily deduce that the force *leads* the velocity in phase, and that the displacement is greater the larger the mass F, as compared to the frictional resistance; in other words, the greater the reactance as compared to the resistance. It may be shown also that the inertia reaction of the same mass F is greater for a higher frequency of oscillation, because the acceleration and retardation are proportionately larger.

Prob. 1. The inductance of a coil is 0.2 henry; its ohmic resistance is negligible. Draw a curve giving the voltage necessary to maintain a current of 12 amp. through the coil, at frequencies ranging from zero to 100 cycles per second.

Ans. A straight line through the origin; at $f = 100$, $E = 1508$ volts.

Prob. 2. A reactive coil without iron draws a current of 75 amp. when connected across a 110-volt 25-cycle circuit. What current would it draw at 60 cycles and at the same voltage, provided that the effect of its resistance can be neglected? Plot a curve of current at intermediate frequencies.

Ans. 31.25 amp.; equilateral hyperbola asymptotic to both axes.

Prob. 3. The reactive magnetizing current of a 2200-volt, 600-kilovolt-ampere, 50-cycle transformer must be not over 2.5 per cent of the full-load current. What is the lower limit of its no-load reactance and inductance?

Ans. 322.5 ohms; 1.027 henrys.

Prob. 4. The coil considered in problem 1 is connected in series with a 100-ohm resistance; it is required to maintain a current of 12 amp. through the two, at various frequencies. Supplement the curve obtained in that problem with curves of voltage drop across the resistance, and the total voltage across the combination. Plot also the corresponding values of power-factor. Determine the ordinates of the curves graphically, and check a few points analytically.

Ans. $E_r = 1200$ volts, independent of the frequency. At $f = 0$, $E_{total} = E_r$, and $\cos\phi = 1$. At $f = 100$, $E_{total} = 1927$ volts, $\cos\phi = 62.25$ per cent.

Prob. 5. Three simultaneous instrument readings in a power house are: 7520 kw.; 66 kv.; 147 amp. The power-factor meter shows that the current is lagging behind the voltage. What are the readings at the same instant at the receiving end of the line, if its resistance is 45 ohms and its reactance 83 ohms. Hint: Draw the vectors of the generator voltage and current in their true relative position. Subtract the ohmic drop in phase with the current, and the reactive drop in quadrature with it. The result will give the receiver voltage in its true magnitude and phase position. Ans. 6547 kw.; 53.4 kv.

Prob. 6. In order to determine the power input into a single-phase 110-volt motor, without the use of a wattmeter, the motor is connected in series with a non-inductive resistance across a 220-volt circuit. The resistance is adjusted so that the voltage across the motor terminals is 110, when the motor is carrying the required load. Under these conditions the voltage across the resistance is found to be 127, and the current through the motor 23 amp. From these data determine graphically the power factor of the motor, and calculate its power input.

Ans. 72.3 per cent; 1826 watts.

Prob. 7. Referring to the preceding problem, calculate $\cos\phi$ trigonometrically, from the triangle of voltages, instead of determining it graphically.

22. Impedance. When a reactance is connected in series with a resistance, eqs. (89) and (91) indicate that the current and voltage relations are determined, not by the value of the reactance alone, but by a composite expression

$$z = \sqrt{r^2 + x^2}. \qquad (92)$$

The quantity z has the dimension of a resistance, and is called the *impedance* of the circuit. It can hardly be called a physical quantity, but rather an abbreviation for a certain combination of the physical properties of a circuit; in other words, an abbreviation for the radical in eq. (92). Introducing the value of z into eqs. (89) and (91), we obtain

$$E = zI \qquad (93)$$

and

$$\cos\phi = r/z. \qquad (94)$$

Impedance may be defined from eq. (93) as the ratio of the voltage to the current in a circuit containing resistance and reactance. In a non-inductive circuit the impedance is simply equal to the total resistance, while in a purely inductive one the impedance is equal to the reactance. It must be clearly understood that eq. (93) gives only the relation between the magnitudes of the vectors. The phase relation is given by Fig. 20, or by eq. (94).

The three quantities r, x, and z form a triangle of which z is the hypothenuse (Fig. 20). This triangle is similar to the triangle of voltages, but the quantities r, x, and z are not vectors in the same sense as currents and voltages are. From the impedance triangle we have the following useful relations:

$$r = z \cos \phi \qquad (95)$$

and

$$x = z \sin \phi. \qquad (96)$$

When two impedances are connected in series (Fig. 22), the voltage and current relations are as represented in Fig. 23. The total terminal voltage E is less than the arithmetical sum of the voltages E_1 and E_2 across the two impedances, and is equal to their geometric sum. The resultant phase angle ϕ has a value intermediate between the phase angles ϕ_1 and ϕ_2 of the two component impedances. It will be seen from the triangle ABC that the resultant voltage

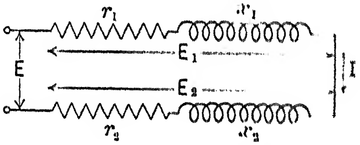

FIG. 22. Two impedances in series.

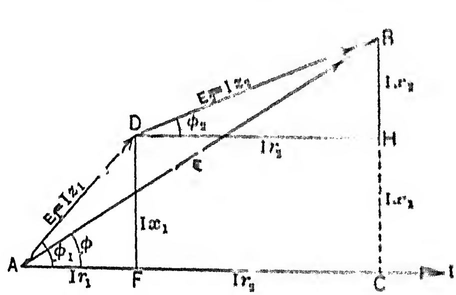

FIG. 23. The current and voltage relations in the circuit shown in Fig. 22.

is the same as that required by an impedance which consists of a resistance $r_1 + r_2$ and a reactance $x_1 + x_2$. In other words,

the resultant impedance is

$$z = \sqrt{(r_1 + r_2)^2 + (x_1 + x_2)^2}, \quad . \quad . \quad . \quad . \quad (97)$$

and the resultant phase angle is determined from the equation

$$\tan \phi = (x_1 + x_2)/(r_1 + r_2). \quad . \quad . \quad . \quad . \quad (98)$$

These equations show that two impedances are added in series by adding the resistances and the reactances separately. An impedance of 5 ohms in series with one of 7 ohms is *not* equal to an impedance of 12 ohms, but as a rule is less. The relations shown in Figs. 22 and 23, and eqs. (97) and (98), are easily extended to any number of impedances in series. Dividing all the voltage vectors in Fig. 23 by the value of the current I, the diagram of voltages is converted into one of impedances, as in Fig. 20, the relations being represented by eqs. (97) and (98). It must be borne in mind, however, that from a physical point of view the latter relations are not vectorial in the same sense as are those of the voltages.

Prob. 1. The impedance of a coil is 7.5 ohms at 60 cycles; the resistance measured with direct current is 6 ohms. What is the inductance?

Ans. 11.9 millihenrys.

Prob. 2. Two impedance coils are connected in series across a 292-volt line. The voltages across the coils are 152 and 175 respectively; the current is 7.3 amp. Knowing that the resistance of the first coil is 10 ohms, determine graphically the resistance of the second; also the impedances of both coils.

Ans. $r_2 = 23.8$; $z_1 = 20.82$; $z_2 = 23.97$, all in ohms.

Prob. 3. When a certain non-inductive resistance is connected across a source of alternating voltage, a current I flows through it. When an inductance, containing negligible resistance, is connected across the same source of voltage, the current is I'. What are the current and the phase displacement when the resistance and the inductance are connected in series across the same source? Solution: Let the unknown voltage be E. The unknown resistance is $r = E/I$; the unknown reactance $x = E/I'$. When the two are connected in series, the impedance $z = [(E/I)^2 + (E/I')^2]^{\frac{1}{2}}$. Consequently, the current is $E/z = II'/(I^2 + I'^2)^{\frac{1}{2}}$; $\tan \phi = x/r = I/I'$.

23. Influence of Inductance with Non-sinusoidal Voltage.

(a) Let an alternating voltage e of an irregular form, such as is shown in Fig. 14, be applied at the terminals of a *pure resistance* r (non-inductive). The current through the resistance is at any instant equal to e/r, and consequently has the same wave form as the voltage.

(b) Let now the same voltage be applied at the terminals of a *pure inductance L* (without resistance). It may be said *a priori* that the current wave will be different from that of the voltage, and will approach more nearly a sine-wave. This follows from the very concept of inductance as the inertia of the circuit; the high-frequency harmonics in the voltage are unable to produce currents of the same magnitude as at lower frequencies, because the reactance offered to each harmonic is proportional to its frequency. This property of an inductance of choking higher harmonics is useful in some applications.

Let the voltage across an inductance be given in the form of a Fourier series,

$$e = E_1 \sin (2\pi ft - \alpha_1) + E_3 \sin 3(2\pi ft - \alpha_3) + \text{etc.}$$

Substituting its value in the fundamental eq. (81), we get

$$E_1 \sin (2\pi ft - \alpha_1) + E_3 \sin 3(2\pi ft - \alpha_3) + \text{etc.} = L\, di/dt.$$

Multiplying both sides of this equation by dt and integrating gives

$$-(E_1/2\pi f) \cos (2\pi ft - \alpha_1) - (E_3/6\pi f) \cos 3(2\pi ft - \alpha_3) - \text{etc.} = Li + \text{const.}$$

The constant of integration is equal to zero, because the current cannot have a unidirectional component without a commutating device or electric valve of some sort. Therefore

$$i = -(E_1/2\pi fL) \cos (2\pi ft - \alpha_1) - (E_3/6\pi fL) \cos 3(2\pi ft - \alpha_3) - \text{etc.}, \quad . \quad . \quad . \quad . \quad . \quad (99)$$

which means that each harmonic in the e.m.f. produces its own current, *as if this harmonic were acting alone*. The total current is the sum of such harmonic currents. The reactance of the coil for the nth harmonic is n times as great as for the fundamental wave; therefore, the higher harmonics in the current are relatively smaller than those in the voltage wave.

(c) Let now a non-sinusoidal alternating voltage be impressed at the terminals of an *impedance coil*, and let it be required to determine the wave form of the current. The result to be expected will be intermediate between those derived for a pure resistance and a pure inductance; *viz.*, the current wave will be more nearly of sine form than the voltage wave, but not to the same extent as in the case of a pure inductance.

Substituting the above given expansion for the voltage wave

into the fundamental eq. (83), we obtain a differential equation for i, which equation some readers may not be able to solve. We choose, therefore, the opposite way; that is, we assume the current wave to be given, instead of the voltage wave, and determine the corresponding voltage wave from eq. (83). This procedure is much simpler, because it involves differentiation instead of integration. Let the current be given in the form

$$i = I_1 \sin (u - \alpha_1) + I_3 \sin 3 (u - \alpha_3) + \text{etc., where } u = 2\pi ft.$$

Substituting this value into eq. (83) and rearranging the terms, gives

$$e = [I_1 r \sin (u - \alpha_1) + 2\pi f L I_1 \cos (u - \alpha_1)] + [I_3 r \sin 3 (u - \alpha_3) + 6\pi f L I_3 \cos 3 (u - \alpha_3)] + \text{etc.} \quad . \quad . \quad . \quad (100)$$

This result shows that each harmonic of the current requires a corresponding harmonic of the voltage, as if it were flowing alone. The total voltage is equal to the sum of the harmonic voltages. Therefore we conclude that, conversely, if the voltage were given, the current would be equal to the sum of the harmonic currents produced by the respective harmonics in the voltage. If the impedance to the first harmonic is $z_1 = \sqrt{r^2 + x^2}$, that to the third harmonic is $z_3 = \sqrt{r^2 + (3x)^2}$, and in general the impedance to the nth harmonic is $z_n = \sqrt{r^2 + (nx)^2}$. The phase displacement between the corresponding harmonics of current and voltage is determined from the condition, $\tan \phi_n = nx/r$, or $\cos \phi_n = r/z_n$.

The general conclusion reached is as follows: When the applied voltage contains higher harmonics, the total current is found by summing the harmonic currents due to each harmonic of the voltage acting alone.

Prob. 1. The effective value of the fundamental wave of an e.m.f. is 110 volts; it has a pronounced third harmonic, of 24 per cent of the fundamental wave. This voltage is applied across a pure reactance, equal to 5 ohms for the fundamental frequency. Calculate the current.

Ans. 22.07 amp.

Prob. 2. An alternating voltage is represented by the expression $170 \sin 250\,t + 62 \sin (1250\,t + 2.3)$. It is applied to an impedance coil having an inductance of 45 millihenrys and a resistance of 7 ohms. Show that the current in amperes is equal to $12.82 \sin (250\,t - 1.015) + 1.09 \sin (1250\,t + 0.853)$.

24. The Extra or Transient Current in Opening and Closing a Circuit. Since an electric current possesses inertia in

the form of inductance, no current can be established or broken instantly, unless the applied electromotive force be infinitely large. Thus, when a large electromagnet is connected to a source of continuous voltage, the current increases during an appreciable interval before it reaches its final value. Again, when the circuit is broken, the current continues in the form of an arc through the air for an appreciable time. In a majority of cases these transient phenomena at the opening and closing of a circuit are of no practical importance, yet there are circumstances under which they must be taken into consideration; for instance, in switching on and off large amounts of energy, in high-frequency oscillations, in highly inductive circuits, etc. We shall consider here two simple cases of such extra currents; namely, when a circuit possessing resistance and inductance is connected to a source of (a) continuous voltage and (b) sinusoidal alternating voltage.

(a) *Direct Voltage.* When e in eq. (83) is constant, one value of i which satisfies this equation is $i = e/r$, because in this case $di/dt = 0$. However, this is not the most general solution, because it is possible to select an exponential expression in addition to the constant i, which will satisfy the equation. Put

$$i = e/r + C\epsilon^{-t/\tau}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

where ϵ is the base of natural logarithms, and C and τ are certain constants. Substituting this value of i into eq. (83), we get

$$Cr\epsilon^{-t/\tau} - (L/\tau)C\epsilon^{-t/\tau} = 0,$$

or

$$\tau = L/r.$$

Besides, $i = 0$ when $t = 0$, so that expression (101) becomes $0 = e/r + C$, from which

$$C = -e/r,$$

and consequently

$$i = (e/r)(1 - \epsilon^{-tr/L}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

In other words, when a direct-current circuit is closed, the current increases at first rapidly, then more and more slowly; and theoretically it reaches its final value of e/r only after an infinite time. In reality, the current becomes practically constant after a fraction of a second, unless the inductance is exceedingly large. The factor $\tau = L/r$ is called the *time constant* of the circuit; it determines the rate of the initial rise in current, and has the dimension of time.

(b) *Sinusoidal Voltage.* If the voltage follows the law $e = E_m \sin 2\pi ft$, one solution of eq. (83), as we have seen before, is $i = (E_m/z) \sin (2\pi ft - \phi)$, where $\cos\phi = r/z$. But this is not the most general solution, because it is possible to add to it an exponential term of the form $C\epsilon^{-t/\tau}$, and to select the time constant τ in such a way that this term will cancel in eq. (83). Since the sine term of the current alone satisfies the equation, we will find as before $\tau = L/r$. The constant C is determined by the condition that $i = 0$ when $t = 0$, or

$$0 = -(E_m/z) \sin\phi + C,$$

from which

$$C = (E_m/z) \sin\phi = E_m x/z^2.$$

Therefore the current

$$i = (E_m/z) \sin (2\pi ft - \phi) + (E_m x/z^2)\epsilon^{-tr/L}. \quad . \quad . \quad (103)$$

Under ordinary conditions the exponential term becomes negligibly small within a fraction of a second, so that it is legitimate to consider the current to be a pure sine-wave, as we have done heretofore. However, the extra current may be of importance in transient phenomena, for instance, at the moment of closing a circuit.

The solutions (102) and (103) of eq. (83) are found above by trials, because it is assumed that the reader is not familiar with the general method for the solution of linear differential equations; otherwise, the solution could have been written directly. Equation (83) is of the form

$$dy/dx + Py = Q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

where P and Q are functions of x or constants. By referring to any book on differential equations, the reader will find that the general solution of this equation is

$$y = \epsilon^{-v}\left[\int \epsilon^{v} Q\,dx + C\right], \quad . \quad . \quad . \quad . \quad (105)$$

where

$$v = \int P\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

Prob. 1. The current in a coil due to a constant e.m.f. reaches 99 per cent of its final value within one hundredth of a second after the circuit is closed. Show that the time constant of the coil is equal to 2.17 milliseconds.

Prob. 2. Show that the time constant may be defined as the interval of time during which the current reaches $(\epsilon - 1)/\epsilon = 0.632$ of its final value.

Prob. 3. Select the constants of an alternating-current circuit so as to have a power-factor of about 80 per cent; and plot curves of (a) the voltage, (b) the sinusoidal component of the current, (c) the exponential component of the current, and (d) the total current, for the first few cycles after the circuit is closed.

Prob. 4. Extend the theory given above to the case where the circuit is closed at an instant when the alternating voltage is not equal to zero.

Prob. 5. Check the solutions (102) and (103), using formula (105).

Prob. 6. When an impedance, consisting of r and L, is suddenly short-circuited, so that e becomes instantly equal to zero, show that the line current gradually disappears according to the exponential law $i = i_0\epsilon^{-rt/L}$, where i_0 is the magnitude of the current at the instant of short-circuit.

CHAPTER VII

SUSCEPTANCE AND ADMITTANCE

25. Concept of Susceptance. The concept of reactance, as introduced in Art. 21, indicates the degree of *difficulty* in forcing an alternating current through a coil, against the reaction of an alternating magnetic field. In this respect, reactance is analogous to resistance. We have seen, however, in Chapter I, that it is more convenient to use conductances, when resistors are connected in parallel. Similarly, when reactive coils or reactors are connected in parallel, it is more convenient in calculations to use the reciprocals of their reactances. The reciprocal of reactance is called *susceptance*, and is usually denoted by the symbol b. Thus, by definition, the susceptance

$$b = 1/x = 1/(2\pi fL). \quad . \quad . \quad . \quad . \quad . \quad (107)$$

By analogy with conductance, one may say that the susceptance measures the degree of *ease* in forcing an alternating current through a coil, against the reaction of a pulsating magnetic field Since reactance is measured in ohms, susceptance is measured in mhos. Equation (87) becomes

$$I = bE, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

it being understood as before that the current lags by 90 degrees behind the voltage. The student is reminded that the concept of susceptance, like that of reactance, implies pure inertia reaction, without any ohmic resistance; this limitation is very important for a clear understanding of the rest of the chapter.

When several inductive coils are connected in parallel, their susceptances are simply added together, or

$$b_{eq} = b_1 + b_2 + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

The proof is similar to that for the addition of conductances (see Art. 3). Thus, a susceptance of 3 mhos in parallel with one of 2 mhos gives a total susceptance of 5 mhos.

Prob. Two reactive coils of 10 and 20 millihenrys respectively are connected, first in series and then in parallel, across a 40-cycle, 180-volt line. The ohmic resistance of the coils is negligible. What is the current in each case? Ans. 23.85 amp.; 107.35 amp.

26. Concept of Admittance. Let now a pure inductance be connected in parallel with a pure ohmic resistance, across a source of alternating voltage E (Fig. 24), and let it be required to find

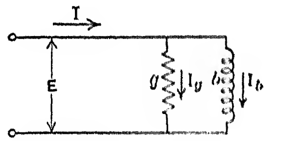

FIG. 24. A susceptance in parallel with a conductance.

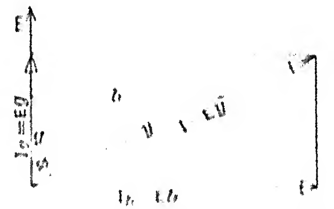

FIG. 25. The voltage and current relations in the circuit shown in Fig. 24.

the total current through the combination. The inductance can be expressed as a susceptance, and the resistance as a conductance. The current through the susceptance, according to eq. (108), is bE, in quadrature with the voltage (Fig. 25); the current through the conductance, according to eq. (2), is gE, in phase with the voltage. The total current

$$I = \sqrt{(Eg)^2 + (Eb)^2} = E\sqrt{g^2 + b^2}, \quad . \quad . \quad . \quad (110)$$

and the phase angle is determined from the relation

$$\tan \phi = Eb/Eg = b/g \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

or

$$\cos \phi = g/\sqrt{g^2 + b^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

In the case of a series connection, we have found it convenient to introduce the impedance z as a symbol for $\sqrt{r^2 + x^2}$. Similarly, in a parallel connection it is convenient to introduce the abbreviation

$$y = \sqrt{g^2 + b^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

The quantity y is called the *admittance* of a circuit, and is measured in mhos, the same as b and g. Equation (110) becomes

$$I = yE, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

and eq. (112),

$$\cos \phi = g/y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

The three quantities g, b and y form a triangle (Fig. 25), in which y is the hypothenuse, and the angle adjacent to g is the phase angle ϕ. From this triangle we obtain two useful relations,

and
$$\left.\begin{aligned} g &= y\cos\phi \\ b &= y\sin\phi. \end{aligned}\right\} \qquad (116)$$

When there are several susceptances and conductances in parallel (Fig. 26), the reactive and the energy components of the current

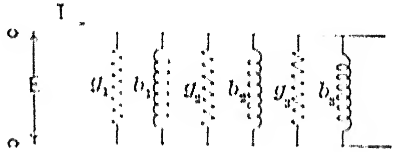

FIG. 26. Susceptances and conductances in parallel.

must be added separately (Fig. 27). Therefore, the amperes per volt in phase or the conductances, and the amperes per volt in

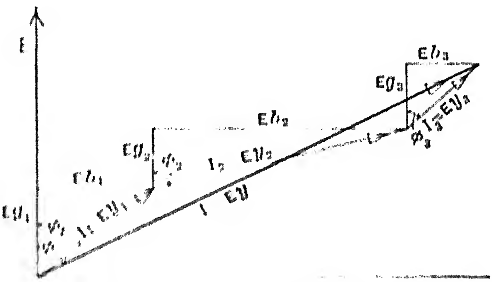

FIG. 27. The voltage and current relations in the circuit shown in Fig. 26.

quadrature or the susceptances, must also be added separately, so that the equivalent admittance

$$y = \sqrt{(g_1 + g_2 + \text{etc.})^2 + (b_1 + b_2 + \text{etc.})^2}, \quad . \quad (117a)$$

and

$$\tan\phi = (b_1 + b_2 + \text{etc.})/(g_1 + g_2 + \text{etc.}). \quad . \quad . \quad . \quad (117b)$$

The student should compare Fig. 27 with Fig. 23 in order to see the similarity of procedure and the difference in the physical phenomena in the two cases. With a series connection, it is the current that is common to all the parts of the circuit, while the partial voltages are added geometrically. In a parallel combination, the voltage is common to all the branches, while the component currents are combined in their proper phase relations.

The following table gives the quantities defined in this and the preceding chapter, in their proper relations.

Friction Effect	Inertia Effect	Connection	Result	Unit
Resistance r	Reactance x	Series	Impedance z	Ohm
Conductance g	Susceptance b	Parallel	Admittance y	Mho

Prob. 1. What susceptance must be connected in parallel with a resistance of 0.2 ohm, in order to bring the power-factor of the combination down to 80 per cent? Also, what is the value of the resultant admittance? Ans. 3.75 mhos; 6.25 mhos.

Prob. 2. Two electrical devices are connected in parallel to a line of voltage E. One device consumes a current I_1 at a power-factor $\cos\phi_1$; the total line current is I, lagging behind the voltage by an angle ϕ. Show how to determine graphically the susceptance and the conductance of both devices.

27. Equivalent Series and Parallel Combinations. Let a resistance r_s be connected in series with a reactance x_s; also let another resistance r_p be connected in parallel with a reactance x_p. If the values of the resistances and reactances are so selected that the series combination, when connected to the same source of supply, will let through the same current at the same power-factor as the parallel combination, then the two combinations are called *equivalent*. It is sometimes convenient to replace a given series combination by an equivalent parallel combination, and vice versa. For instance, when some parts of a circuit are in parallel and others in series, it is convenient for numerical calculations to replace them all by an equivalent parallel or series combination.

The problem is to find the relation between the four quantities r_s, r_p, x_s and x_p, if these quantities form two equivalent combinations. According to the above-given definition, the angle ϕ is the same for both, and besides, according to eqs. (93) and (114),

$$y = 1/z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

where y refers to the parallel combination and z to the equivalent series combination. Combining now eqs. (116), (95) and (96), we have

$$1/r_p = g = y\cos\phi = (1/z)\,(r_s/z) = r_s/z^2;$$
$$1/x_p = b = y\sin\phi = (1/z)\,(x_s/z) = x_s/z^2;$$

or

$$r_s r_p = z^2 = 1/y^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

$$x_s x_p = z^2 = 1/y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

By means of eqs. (119) and (120) a series combination can be replaced by an equivalent parallel combination, and vice versa. Instead of r_p and x_p, their reciprocals, g and b, may be used. In practice, g and b are usually spoken of as the conductance and the susceptance of either the series or the parallel combination; but it must be clearly understood that they are the reciprocals of r_p and x_p, and not of r_s and x_s. If r_s and x_s are given, it is first necessary to determine r_p and x_p from eqs. (119) and (120), and then to take their reciprocals. In other words, for a series circuit the equivalent conductance and susceptance are

$$g = r_s/z^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

and

$$b = x_s/z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

On the other hand, if g and b are given,

$$r_s = g/y^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

$$x_s = b/y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

The reciprocals of r_s and x_s are of no practical importance, and are not used in this work.

Prob. 1. An impedance coil has a reactance of 7.5 ohms; the resistance of the winding is 2 ohms. What are the susceptance and the conductance of the equivalent parallel combination?

Ans. 124.3 and 33.2 millimhos.

Prob. 2. Check the answer to the foregoing problem by actually calculating the current and the power-factor of the series and the parallel combinations at some assumed voltage.

Prob. 3. Show that r_p and x_p are always larger than r_s and x_s respectively. Hint: In eqs. (119) and (120) replace z^2 by $r_s^2 + x_s^2$, and solve for r_p and x_p.

Prob. 4. An apparatus takes 25 amp. and 2000 watts at 110 volts, the current being a lagging one. What are the equivalent conductance and susceptance of the device? What are the resistance and reactance in series equivalent to this apparatus?

Ans. 0.165 mho; 0.156 mho; 3.2 ohms; 3.04 ohms.

Prob. 5. In adjusting a measuring instrument, a non-inductive resistance of 120 ohms was used in parallel with a choke coil. The impedance of the coil was 75 ohms, its resistance 16 ohms. In the regular manufacture of the instrument it is desired to use a resistance and a reactance in series. Determine their values, either graphically or analytically. Ans. $r_s = 38.0$ ohms; $x_s = 44.3$ ohms.

28. Impedances in Parallel and Admittances in Series. In the preceding chapter we have learned how to add impedances in series, and in this chapter how to add admittances in parallel. Let now two or more impedances be connected in parallel, and let it be required to find the equivalent impedance. This is done by replacing each of the given impedances by an equivalent parallel combination, and then adding their admittances in parallel, according to the rule developed above. Conversely, let several admittances be connected in series, and let it be required to find the equivalent admittance. To solve this problem, each parallel combination is replaced by an equivalent series combination, and then the impedances are added in series. The student understands, of course, that the addition in both cases is geometric, and that only like components can be added algebraically. Problems of this kind occur, for instance, in the theory of transmission lines, transformers, and induction motors; for this reason it is important that the student understand the equivalent combinations, and that he acquire facility in changing from a series to a parallel combination, and vice versa, as is explained in the preceding article.

Prob. 1. The load of a single-phase, 6600-volt generator is estimated to consist of 1200 kw. of lamps, practically non-inductive, and of 800 kw. of motors, working at an average power-factor of 75 per cent. What will be the expected generator output, in amperes, and the power-factor? Solution: The energy component of the motor current is $800/6.6 = 121.2$ amp.; the reactive component is $121.2 \tan \phi = 106.8$ amp. The lamp current is $1200/6.6 = 181.8$ amp. The total energy component of the generator current is $121.2 + 181.8 = 303$ amp. Consequently, the total generator current is $(303^2 + 106.8^2)^{\frac{1}{2}} = 321.3$ amp.; the power-factor is $303/321.3 = 94.3$ per cent.

Prob. 2. Check the solution of the preceding problem graphically.

Prob. 3. Three resistances of 2, 5 and 10 ohms, and two reactances of 4 and 2.5 ohms, are all connected in parallel across a 250-volt alternating-current line. What are the total current and the power-factor of the combination? *Ans.* 258 amp.; 77.5 per cent.

Prob. 4. Three impedance coils, having ohmic resistances of 2, 3 and 4 ohms respectively, and inductances of 13, 10 and 22 millihenrys, are connected in parallel across a source of 220-volt, 60-cycle alternating voltage. Calculate the total current and the power-factor. Check the solution graphically. *Ans.* 110 amp.; $\cos \phi = 0.495$.

Prob. 5. Solve the preceding problem for a frequency of 25 cycles per second. Construct the vector diagrams of the currents in both problems to the same scale, so as to see the influence of the frequency.

Prob. 6. In problem 4, let the total current be given in magnitude, but not in its phase position; assume the inductance of the third coil to be unknown. Show how to determine analytically and graphically the vector of the current in the third coil, and the position of the vector of the total current.

Prob. 7. The admittance of a winding is 0.2 mho; the current through the winding lags by 34 degrees with respect to the voltage at its terminals. Determine the resistance and the reactance of the winding.

Ans. 4.145 ohms; 2.796 ohms.

Prob. 8. A coil having a resistance of 2.3 ohms and a reactance of 5 ohms is connected in parallel with another coil, for which $r = 3$ ohms and $x = 4$ ohms. Calculate the resistance and the reactance of the equivalent series circuit. Ans. 1.36 ohms; 2.255 ohms.

Prob. 9. The coils given in the preceding problem are connected in parallel across 55 volts. Calculate the total current, its energy and reactive components, and the power-factor of the combination.

Ans. 20.85 amp.; 10.78 amp.; 17.88 amp.; $\cos \phi = 0.5165$.

CHAPTER VIII

THE USE OF COMPLEX QUANTITIES

29. Addition and Subtraction of Projections of Vectors. With the explanation given in the preceding four chapters, the student is enabled to handle, by means of vector diagrams, problems involving resistances and reactances in alternating-current circuits. A number of problems in transmission-line calculations and in the theory of alternating-current machinery may be solved by the use of such vector diagrams. The disadvantages of the graphical method are: (1) Results are usually obtained which hold for one specific case only; an analysis of the effect of various factors is often difficult. (2) Some vectors may be many times smaller than others; for instance, the voltage drop in a transmission line, as compared to the line voltage itself. Therefore, the diagram must be drawn to a very large scale, or else the results are not sufficiently accurate. In addition to these drawbacks, some engineers object to graphical methods in general, as involving the use of drawing instruments, which may not be convenient.

On the other hand, vector diagrams are quite convenient in some practical cases; moreover, they are helpful for the understanding of general relations in a circuit, without reference to particular numerical values. Again, in some problems, the unknown vectors can be calculated from the vector diagram trigonometrically, without the necessity of actually drawing it to scale.

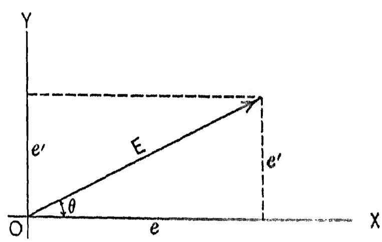

FIG. 28. A vector and its projections.

It is possible to treat vectors analytically, using their projections on two axes, as in analytic geometry (Fig. 28). A vector, such as E, can be defined either by its magnitude and phase angle θ, or by its projections e and e' upon the axes of coördinates.

If E and θ are given, the projections are calculated from the expressions

$$\left.\begin{aligned} e &= E\cos\theta; \\ e' &= E\sin\theta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

If the projections are given, the vector itself is determined in magnitude and position from the equations

$$E = (e^2 + e'^2)^{\frac{1}{2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

$$\tan\theta = e'/e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

In numerical computations it is more convenient first to calculate $\tan\theta$ from eq. (127) and then to determine E from one of the eqs. (125), using trigonometric tables. This does away with the necessity for squaring the projections and extracting a square root.

The fact that e and e' are components of the vector E along two perpendicular axes is expressed *symbolically* thus:

$$\dot{E} = e + je'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

Here j is a symbol which indicates that the projection e' refers to the vertical axis. This symbol must not have any *real* value; for the time being, it may be considered merely as an abbreviation of the words "along the vertical axis." The sign plus in eq. (128) denotes the geometric addition. The dot under E signifies that by $\dot{E}$ is meant not only the magnitude of the vector, but its direction as well, the latter being defined by the projections. When the magnitude only is meant, the dot is omitted.

The foregoing notation has been introduced by Dr. Charles P. Steinmetz, and is now universally used in this country. Much credit is also due to Dr. Steinmetz for developing the analytic method, used below, of dealing with alternating currents and voltages by means of their projections.

The addition and subtraction of vectors are reduced simply to the addition and subtraction of projections. According to Fig. 12, the projection of a vector on any axis is equal to the sum of the corresponding projections of its component vectors on the same axis. Thus, if a current is represented as a vector by its projections $50 + j\,70$ amp., and another current by $100 + j\,40$ amp., the vector sum of these currents is $150 + j\,110$ amp. Or, the resultant of two voltages, $\dot{E}_1 = e_1 + je_1'$ and $\dot{E}_2 = e_2 + je_2'$, is

$$\dot{E}_{eq} = \dot{E}_1 + \dot{E}_2 = (e_1 + e_2) + j(e_1' + e_2').$$

As an illustration, let us solve problem 3, Art. 14, by the method of projections. Take the voltage vector of the first alternator in the horizontal direction, this being the simplest assumption. This vector is therefore expressed as $\dot{E}_1 = 2300 + j\,0$. The horizontal projection of the second vector is $1800 \cos 27° = 1603.8$ volts, and its vertical projection is $1800 \sin 27° = 817.2$ volts. Both of these projections are positive, because the second vector leads the first, and is therefore in the first quadrant. Thus, $\dot{E}_2 = 1603.8 + j\,817.2$ volts. The resultant voltage, $\dot{E}_{eq} = \dot{E}_1 + \dot{E}_2 = 3903.8 + j\,817.2$ volts. For some purposes, it is sufficient to leave the answer in this form; if, however, the magnitude and phase position are required, they are found as explained above: $\tan\theta = 817.2/3903.8 = 0.2092$; $\theta = 11° 49'$; $\cos\theta = 0.9786$; $E_{eq} = 3903.8/0.9786 = 3988$ volts.

If the terminals of the second machine be reversed, then $\dot{E}_{eq} = \dot{E}_1 - \dot{E}_2 = 696.2 - j\,817.2$. This vector has a positive horizontal projection and a negative vertical projection. Consequently, it lies in the fourth quadrant, and lags behind the reference vector by less than 90 degrees. Proceeding as above, we find $\theta = -49° 32'$; $E_{eq} = 1074$ volts.

Prob. 1. Solve problem 1, Art. 14, by the method of projections, assuming the vector of the first current to be horizontal.

Prob. 2. Check the solution of problem 4, Art. 14, by the method of projections.

30. Rotation of Vectors by Ninety Degrees. In problems involving reactance, it is necessary to multiply the vector of the current by the reactance of the circuit and then turn it by 90 degrees, in order to determine the reactive drop in voltage. The simple multiplication of the vector of current by the reactance converts it into a vector of voltage, and thus merely changes the scale. But turning the vector modifies the relative magnitudes of its projections; it is, therefore, necessary to find a relation between the magnitudes of the original and the new projections.

In the simplest case let a vector $\dot{E}_1$ be drawn along the reference axis, or axis of abscissæ, and let its length be a. In the symbolic notation it is represented as $\dot{E}_1 = a$, the other projection being zero. After having been turned by 90 degrees counter-clockwise, the vector is directed along the positive axis of ordinates, and is symbolically represented as $\dot{E}_2 = ja$, the horizontal pro-

jection being zero. Thus, in this particular case, a rotation by 90 degrees is equivalent to a multiplication by j.

It is convenient to define j in such a manner that multiplication of any vector by j will turn the vector by 90 degrees in the positive direction (counter-clockwise), *while division by j will turn the vector by 90 degrees in the negative direction* (clockwise). In order to find a value of j which satisfies these requirements, let the vector E_2 be turned again by 90 degrees counter-clockwise, being now directed along the negative X-axis. Its expression is now $E_3 = -a$. On the other hand, the same expression must be obtained by multiplying E_2 by j. Therefore, we have $-a = j^2a$, or $j^2 = -1$; consequently, $j = \sqrt{-1}$. If the original vector E_1 is to be turned by 90 degrees clockwise, we must, according to our assumption, divide it by j. We then have $E_4 = a/j$, or, multiplying the numerator and the denominator by j, $E_4 = ja/j^2$. If $j^2 = -1$, as it appears to be above, then $E_4 = -ja$. This checks with the preceding result, because $E_2 = -E_4$. It will thus be seen that the value of $j^2 = -1$ satisfies the requirements set above, when the original vector is directed along one of the axes of coördinates.

Let now the original vector E_1 (Fig. 29) have an arbitrary direction in the first quadrant, or $E_1 = a + jb$. Multiplying E_1 by j we must get the vector E_2, of the same magnitude, but in the second quadrant and perpendicular to E_1. E_2 has a vertical projection equal to the horizontal projection a of the original vector E_1; the horizontal projection of E_2 is negative, and is equal in its absolute value to the vertical projection b of the vector E_1. Thus, the new vector is expressed as $E_2 = -b + ja$. On the other hand, multiplying E_1 by j we have $jE_1 = ja + j^2b = ja - b$, which is the same as above. Therefore, in this case also the assumption $j^2 = -1$ is correct, and leads to rotation by 90 degrees. It is left to the student to verify the cases in which the vector lies in some other quadrant, and where E_1 is divided by j, for rotation by 90 degrees in the negative direction.

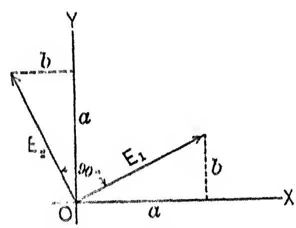

FIG. 29. The relation between the projections of two vectors, perpendicular to each other.

Expressions of the form $a + jb$, where a and b are real quantities and $j = \sqrt{-1}$, are called in algebra *complex* quantities.

The student need not be discouraged by the name, because for our purposes j is simply a quantity which separates the two projections of a vector, obeys the law of multiplication and division, and is of such a nature that $j^2 = -1$. Moreover, solutions by means of complex quantities are quite as simple as by other methods.

Prob. 1. A current of $80 + j\,43$ amp. flows through a resistance of 2 ohms in series with a reactance of 3 ohms. Find the voltage drop across the impedance. Solution: The vector of the voltage consists of two components, representing the ohmic and the reactive drop respectively. The ohmic drop, $\dot{E}_1$, is equal to $2\,(80 + j\,43) = 160 + j\,86$ volts. To find the inductive drop, $\dot{E}_2$, the vector of the current must be multiplied by $x = 3$, and then turned by 90 degrees, in other words, multiplied by j. Thus, $\dot{E}_2 = 3\,j\,(80 + j\,43) = -129 + j\,240$ volts. The total voltage $\dot{E} = \dot{E}_1 + \dot{E}_2 = 31 + j\,326$ volts.

Prob. 2. Solve the preceding problem when the voltage is given and the current is unknown. *First Solution:* Let the unknown current be represented by its projections as $i + ji'$. We have, as in the preceding problem,

$$2\,(i + ji') + 3\,j\,(i + ji') = 31 + j\,326, \quad . \quad . \quad . \quad (129)$$

or, collecting the terms containing j,

$$(2\,i - 3\,i') + j\,(2\,i' + 3\,i) = 31 + j\,326. \quad . \quad . \quad . \quad (130)$$

This equation can be satisfied only if the terms with and without j are equal to each other respectively, because a real quantity cannot be equal to an imaginary one. Or, from a geometric point of view, the left-hand side and the right-hand side of eq. (130) each represent a vector by its projections. But two vectors are identical only when their corresponding projections are equal. Thus, we have

$$2\,i - 3\,i' = 31;\ 2\,i' + 3\,i = 326.$$

Solving these equations for i and i', we find $i = 80$, $i' = 43$, as in the preceding problem. *Second Solution:* Equation (129) can be written in the form $(2 + 3\,j)\,(i + ji') = 31 + j\,326$; or, $i + ji' = (31 + j\,326)/(2 + 3\,j)$. Considering here j as an ordinary algebraic quantity, we can get rid of it in the denominator by multiplying both the numerator and the denominator by $2 - 3\,j$. The result is

$$i + ji' = \frac{(31 + j\,326)\,(2 - 3\,j)}{2^2 - (3\,j)^2} = \frac{1040}{13} + j\,\frac{559}{13},$$

or $i + ji' = 80 + j\,43$, as before.

Prob. 3. A voltage of $28 + j\,120$ volts applied to the terminals of a coil produces in it a current equal to $4 + j\,1.5$ amp. Determine the resistance and the reactance of the coil.

Ans. $r = 16$ ohms; $x = 24$ ohms.

Prob. 4. Verify the answer to problem 4, Art. 28, by the method of projections, assuming the vector of the voltage to be horizontal.

31. Impedance and Admittance Expressed as Complex Quantities or Operators. Let it be required to find the voltage necessary to maintain a current $i + ji'$ through a resistance r and reactance x in series. The voltage drop in the resistance is $r\,(i + ji')$; that in the reactance is $jx\,(i + ji')$. Hence, the total voltage is $\dot{E} = r\,(i + ji') + jx\,(i + ji')$, or

$$\dot{E} = e + je' = (r + jx)\,(i + ji'). \quad . \quad . \quad . \quad (131)$$

It is legitimate to factor out the expression $(i + ji')$, and to treat j as any other algebraic quantity, because j is now assigned a definite value, $\sqrt{-1}$. Moreover, eq. (131) represents simply the geometric addition of four component vectors, two of them directed along the X-axis and the other two along the Y-axis. As long as this interpretation is kept in mind, the terms may be arranged in any desired order.

Equation (131) shows that, in order to obtain the expression of the voltage drop through an impedance, the current must be multiplied by the complex quantity $r + jx$. The expression $r + jx$ is not a vector, because it does not stand for a sine-wave, but an *operator* upon the vector of the current. The operation consists, first, in multiplying the vector of the current by r, then in multiplying the same vector by x and turning it by 90 degrees in the positive direction, and finally, in adding the two vectors geometrically. All these operations are included in the expression $r + jx$, which is called the *impedance operator*.

In order to get the projections e and e' of $\dot{E}$ from eq. (131), the terms on the right-hand side must be actually multiplied and the results represented in the form of a complex quantity. We get then, separating the real and the imaginary parts,

$$\dot{E} = e + je' = (ri - xi') + j\,(ri' + xi).$$

The real and the imaginary parts on each side of this equation are equal to each other respectively, because they represent the projections of the same vector $\dot{E}$ upon the two axes. Consequently

$$\left.\begin{aligned} e &= ri - xi'; \\ e' &= ri' + xi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

In problems these steps are best left until the numerical values have been substituted, in order to avoid complicated expressions.

If the voltage and the impedance are given, and it is required to find the current, we get from eq. (131) the relation

$$i + ji' = (e + je')/(r + jx).$$

In order to reduce the right-hand side of this equation to the form of a complex quantity, we multiply the numerator and denominator by the expression $r - jx$. This gives

$$i + ji' = \frac{(e + je')(r - jx)}{r^2 - (jx)^2} = \frac{re + xe'}{r^2 + x^2} + j\frac{re' - xe}{r^2 + x^2}. \quad (133)$$

Equating the real and the imaginary parts respectively, we obtain

$$\left.\begin{aligned} i &= (re + xe')/z^2; \\ i' &= (re' - xe)/z^2. \end{aligned}\right\} \quad \ldots \ldots \quad (134)$$

Equation (131) expresses the fact that the voltage $\dot{E}$ is equal to the product of the current by the impedance, if the operator $(r + jx)$ be considered as the impedance of the circuit in the complex notation. Denote the impedance by Z, then

$$Z = r + jx. \quad \ldots \ldots \quad (135)$$

Here capital Z is used to indicate that it is a complex quantity, as distinguished from the numerical value z of the same impedance. The letter is not provided with a dot, because Z is not a vector, but an *operator*.

In the abbreviated notation, eq. (131) becomes

$$\dot{E} = \dot{I}Z. \quad \ldots \ldots \quad (136)$$

In this expression each letter stands for a complex quantity, so that when actual numerical or algebraic relations are necessary, the expression must again be expanded into (131) and the multiplication of the two complex quantities actually performed.

Instead of *dividing* the voltage by the operator $(r + jx)$ and then eliminating j from the denominator, it is more convenient to introduce another operator by which the voltage must be *multiplied* in order to obtain the current. It will be remembered from Art. 26, that a voltage must be multiplied by an admittance in order to get the current. Consequently the operator in question must be expected to have the elements and the dimensions of an admittance. Replacing the given series combination by an equivalent parallel combination (Art. 27), the unknown current is split into a component $\dot{E}g$ through a pure conductance, and a component $-j\dot{E}b$ through a pure susceptance in parallel with the con-

ductance. The latter component is provided with the prefix $-j$, because it lags by 90 degrees behind the voltage. Thus, the total current

$$\dot{I} = \dot{E}\,(g - jb). \quad \ldots \ldots \ldots \quad (137)$$

The expression

$$Y = g - jb \,. \quad \ldots \ldots \ldots \ldots \quad (138)$$

is called the *admittance operator*. The symbol Y, like the symbol Z above, is not provided with a dot because it is not a vector. Combining the two preceding equations gives

$$\dot{I} = Y\dot{E}. \quad \ldots \ldots \ldots \ldots \quad (139)$$

Equations (136) and (139) represent generalized forms of Ohm's law for alternating currents, corresponding to the simple expressions (1) and (2) for direct current. Equation (139) is an abbreviated form of the relation

$$i + ji' = (e + je')\,(g - jb). \quad \ldots \ldots \quad (140)$$

Multiplying out and equating the real and the imaginary parts on both sides of this equation, we get

$$\left.\begin{aligned} i &= ge + be'; \\ i' &= -be + ge'. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (141)$$

The relations between r, x and z on one hand, and g, b and y on the other, are deduced in Art. 27; it being understood, of course, that in the present treatment the two combinations are equivalent. Equations (136) and (139) imply that

$$YZ = 1, \quad \ldots \ldots \ldots \ldots \quad (142)$$

or

$$(r + jx)\,(g - jb) = 1.$$

Substituting into this last equation the values of g and b from eqs. (121) and (122) and performing the multiplication, it will be found that the equation is reduced to the identity $1 = 1$, this being a check on eq. (142).

With the abbreviated notation of complex quantities, using the symbols $\dot{E}$, $\dot{I}$, Z and Y, alternating-current problems are solved almost as easily as direct-current problems. Either the impedance operator or the admittance operator is used, depending upon the relative connection of the parts of the circuit, whether parallel or series. In many cases the abbreviated notation may be preserved until the solution has been obtained, the projections of the vectors, $e + je'$ and $i + ji'$, and the expanded forms of the

operators, $r + jx$ and $g - jb$, only then being substituted in a numerical form to get the final answer.

Prob. 1. Find, by means of complex quantities, the voltage required in problem 5, Art. 21. Solution: Assume the vector of the current to be the reference vector. At the power house $\cos\phi = 7520/(66 \times 147) = 0.775$; $\sin\phi = 0.632$. The generator voltage is $66 \times 0.775 + j\,66 \times 0.632 = 51.15 + j\,41.71$ kilovolts. The voltage drop in the line is $147(45 + j\,83) = 6615 + j\,12{,}200$ volts. Hence, the load voltage is $(51.15 - 6.61) + j(41.71 - 12.2) = 44.54 + j\,29.51$ kilovolts. The numerical value of the load voltage is $\left(\overline{44.54}^2 + 29.51^2\right)^{\frac{1}{2}} = 53.43$ kilovolts.

Prob. 2. Determine analytically the resistance r_2 required in problem 2, Art. 22.

Prob. 3. A voltage equal to $180 + j\,75$ produces a current of $7 + j\,1.5$ amp. What is the impedance of the circuit?

Ans. $26.78 + j\,4.97$ ohms.

Prob. 4. Power is transmitted from a single-phase alternator to a load consisting of a resistance of 1.17 ohms in series with a reactance of 0.67 ohm. The generator voltage is 2300, and the impedance of the transmission line is $0.085 + j\,0.013$ ohm. Determine (a) the line current; (b) the voltage drop in the line; (c) the receiver voltage. Take the generator voltage as the reference vector.

Ans. (a) $1413.6 - j\,769.6$ amp.; (b) $130.1 - j\,47$ volts; (c) $2169.9 + j\,47$ volts. Use the admittance operator to obtain the current, and the impedance operator to calculate the line drop.

Prob. 5. A voltage, $e + je'$, is impressed across the impedances $r_1 + jx_1$ and $r_2 + jx_2$ in parallel. Find the total current. Solution: The total conductance is $g = r_1/z_1^2 + r_2/z_2^2$, and the total susceptance is $b = x_1/z_1^2 + x_2/z_2^2$. Hence, the current $i + ji' = (e + je')(g - jb) = (eg + e'b) + j(e'g - eb)$.

Prob. 6. Extend the solution of the preceding problem to the case in which more than two impedances are in parallel.

Ans. $i + ji' = [e\Sigma(r/z^2) + e'\Sigma(x/z^2)] + j[e'\Sigma(r/z^2) - e\Sigma(x/z^2)]$.

Prob. 7. Two impedances, $r_1 + jx_1$ and $r_2 + jx_2$, in parallel, are connected in series with a third impedance $r + jx$. Show how to determine the total voltage, knowing the total current $i + ji'$; or, how to find the expression for the total current when the total voltage $e + je'$ is given.

Prob. 8. Show how to solve the preceding problem when both the current and the voltage are given, but either the impedance $r_1 + jx_1$ or the impedance $r + jx$ is unknown.

CHAPTER IX

THE USE OF COMPLEX QUANTITIES—(*Continued*)

32. Power and Phase Displacement Expressed by Projections of Vectors. Let an alternator supply a current $I = i + ji'$ at a voltage $E = e + je'$, and let it be required to calculate the power output of the generator. The expression for the average power is $P = EI \cos \phi$, where ϕ is the phase displacement between E and I. The angle ϕ is the difference between the angles θ_e and θ_i which the vectors E and I respectively form with the reference axis. Hence, we have

$$\begin{aligned} P &= EI \cos \phi = EI \cos (\theta_e - \theta_i) \\ &= E \cos \theta_e \cdot I \cos \theta_i + E \sin \theta_e \cdot I \sin \theta_i. \end{aligned}$$

Remembering that $E \cos \theta_e$, $E \sin \theta_e$, etc., represent the projections of the given vectors on the axes of coördinates, we have simply

$$P = ei + e'i'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

Another way of deducing expression (143) is to resolve the given vectors of current and voltage into their components along the axes of coördinates, and to consider the contribution of each projection to the total power. The projections e and i, being in phase, give the power ei. Similarly, the projections e' and i' give the power $e'i'$. The projection i' of the current gives zero average power with the projection e of the voltage, the two being in phase quadrature. For the same reason the average power resulting from e' and i is equal to zero. Thus, $ei + e'i'$ represents the total average power.

To find the phase displacement, or the power-factor of the output, we write

$$\tan \phi = \tan (\theta_e - \theta_i) = \frac{\tan \theta_e - \tan \theta_i}{1 + \tan \theta_e \tan \theta_i},$$

or

$$\tan \phi = \frac{(e'/e) - (i'/i)}{1 + (e'/e) \cdot (i'/i)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

Knowing $\tan \phi$, its cosine is found from trigonometric tables.[1]

[1] Or else the power-factor, $\cos \phi = \cos (\theta_e - \theta_i)$, can be found from the relations $\theta_e = \tan^{-1} e'/e$, and $\theta_i = \tan^{-1} i'/i$.

Power-factor can also be determined directly from the expression

$$\cos\phi = P/EI = (ei + e'i')/[(e^2 + e'^2)(i^2 + i'^2)]^{\frac{1}{2}}, \qquad (145)$$

but the calculations are more involved than when formula (144) is used.

The power calculated by means of formula (143) sometimes comes out negative, if some of the projections of E and I are negative. The interpretation is that the phase displacement between the current and the voltage is over 90 degrees, so that power is being supplied to the machine, instead of being delivered by it. In other words, the machine acts as a motor and not as a generator. Tan ϕ in formula (144) may also be negative, which means either that the current is leading, or that it is lagging by an angle larger than 90 degrees. The question is decided by reference to the sign of the power.

For the reactive power (Art. 19) we have

$$P_r = EI\sin\phi = EI\sin(\theta_e - \theta_i) = E\sin\theta_e I\cos\theta_i - E\cos\theta_e I\sin\theta_i,$$

or

$$P_r = e'i - ei'. \qquad (146)$$

The apparent power is

$$P_a = (i^2 + i'^2)^{\frac{1}{2}}(e^2 + e'^2)^{\frac{1}{2}}. \qquad (147)$$

However, it is sometimes more convenient to determine the apparent power from the relation

$$P_a = P/\cos\phi, \qquad (148)$$

where P is calculated from eq. (143), and $\cos\phi$ is found from trigonometric tables, knowing $\tan\phi$ from eq. (144).

Prob. 1. The terminal voltage of an alternator is $5370 + j\,735$; the line current is $173 - j\,47$ amp. Calculate the output of the machine and the power-factor of the load. Ans. 894.5 kilowatts; 92 per cent.

Prob. 2. In the preceding problem, what must be the projection of the current upon the Y-axis in order that the power shall become zero?
Ans. $i' = -1264$ amp.

Prob. 3. Let the line current in problem 1 be $-58 + j\,12$ amp. Explain the negative sign of the power and the plus sign of tan ϕ. Draw the vectors of the current and voltage.

Prob. 4. A synchronous machine generates a voltage equal to $2300 - j\,50$ volts, and supplies a current, through an impedance of $5 + j\,50$ ohms, to another synchronous machine generating a counter-e.m.f. of $2300 + j\,50$ volts. What is the power output of the first machine? Is the current leading or lagging? Make clear to yourself the physical meaning of the answer. Ans. -4.55 kw.; $\phi = 173°\,3'$ lagging.

Prob. 5. A current of $350 - j\,75$ amp. is maintained through an impedance, the power output being 952 kw. at a power-factor of 86 per cent lagging. Find the voltage across the impedance. Hint: Solve eqs. (143) and (144) together for the unknown projections e and e'.

Ans. $2930 + j\,987$ volts.

Prob. 6. Solve problem 5 by calculating the value of the impedance, and multiplying the impedance by the current. Hint: power $= I^2r$; $x = r \tan \phi$.

Prob. 7. Solve problem 5, using the expression for the reactive power.

33. Vectors and Operators in Polar Coördinates. Instead of representing a vector by its orthogonal projections, as in eq. (128), it is sometimes more convenient to express the same vector as a complex quantity in terms of its magnitude and direction. Substituting the values of e and e' from eqs. (125) into eq. (128), we obtain

$$\underset{\cdot}{E} = E\,(\cos\theta + j\sin\theta). \quad . \quad . \quad . \quad . \quad . \quad (149)$$

Similarly, a current in phase with this voltage is expressed as

$$\underset{\cdot}{I} = I\,(\cos\theta + j\sin\theta), \quad . \quad . \quad . \quad . \quad . \quad (150)$$

while a current lagging by an angle ϕ behind the voltage $\underset{\cdot}{E}$ is represented by the equation

$$\underset{\cdot}{I} = I\,[\cos(\theta - \phi) + j\sin(\theta - \phi)]. \quad . \quad . \quad . \quad (151)$$

When the vectors of currents and voltages are expressed in the trigonometric form shown above, it is convenient to use the operators Z and Y in a similar form. Substituting the values of r and x from eqs. (95) and (96) into eq. (135), we get

$$Z = z\,(\cos\phi + j\sin\phi). \quad . \quad . \quad . \quad . \quad . \quad (152)$$

In a similar manner, using eqs. (116) in eq. (138), gives

$$Y = y\,(\cos\phi - j\sin\phi). \quad . \quad . \quad . \quad . \quad . \quad (153)$$

When calculating the voltage drop $\underset{\cdot}{I}Z$ or the current $\underset{\cdot}{E}/Z$ it is necessary to find the product or the ratio of two complex expressions of the form $\cos\theta + j\sin\theta$. By actually performing the multiplication and separating the real from the imaginary term we find that

$$(\cos\theta + j\sin\theta)\,(\cos\phi + j\sin\phi) = \cos(\theta + \phi) + j\sin(\theta + \phi). \quad (154)$$

This gives a simple rule for the multiplication of two or more complex quantities in the trigonometric form. In order to deduce

a similar rule for division, we observe that

$$1/(\cos\phi + j\sin\phi) = \cos\phi - j\sin\phi = \cos(-\phi) + j\sin(-\phi). \quad . \quad . \quad (155)$$

This relation is easily verified by multiplying the numerator and the denominator of the left-hand side of the equation by $\cos\phi - j\sin\phi$, so as to get rid of the complex quantity in the denominator. Equation (155) leads to the following rule for the division of complex quantities in trigonometric form:

$$(\cos\theta + j\sin\theta)/(\cos\phi + j\sin\phi) = \cos(\theta - \phi) + j\sin(\theta - \phi). \quad (156)$$

Thus, for instance, if the current given by eq. (150) flows through an impedance expressed by eq. (152), the required terminal voltage is

$$\dot{E} = \dot{I}\dot{Z} = Iz[\cos(\theta + \phi) + j\sin(\theta + \phi)], \quad . \quad (157)$$

which result simply means that the voltage is equal to Iz and leads the current by the angle ϕ.

The operator given by eq. (152) multiplies a vector by z and turns it by the angle ϕ in the positive direction. Hence, the operator $(\cos\phi + j\sin\phi)$ simply turns a vector by the angle ϕ, without changing its length. Thus, if it be required to turn a vector $\dot{A} = a + ja'$ by an angle α in the positive direction, the projections of the new vector are found from the following expression:

$$(a + ja')(\cos\alpha + j\sin\alpha) = (a\cos\alpha - a'\sin\alpha) + j(a'\cos\alpha + a\sin\alpha). \quad . \quad . \quad . \quad . \quad (158)$$

Of course, the same result could be obtained by first calculating the angle θ which the vector $\dot{A}$ forms with the reference axis, from the relation $\tan\theta = a'/a$, and then determining the new projections $A\cos(\theta + \alpha)$ and $A\sin(\theta + \alpha)$.

Voltage Regulation of a Transmission Line.[1] As an example of the use of complex quantities in the trigonometric form, let us consider the voltage regulation of a single-phase transmission line. Let the resistance and reactance of the line, and the generator voltage E_1, be given; and let it be required to determine the receiver voltage E_2 for a given current I and a given power-

[1] The electrostatic capacity of the line is disregarded here; a complete treatment of the regulation of a transmission line, taking into account the capacity and leakage, is given in Arts. 68 and 69 at the end of the book.

factor of the load $\cos \phi'$. In the symbolic notation we have

$$\dot{E}_1 = \dot{E}_2 + \dot{I}Z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

where the impedance Z of the line is known, and is expressed by eq. (152). The phase angle ϕ refers to the line, the angle ϕ' to the load.

When actually solving an equation such as (159), it is highly important to select the reference axis in the most advantageous way, so as to simplify the calculations as much as possible. In the case under consideration, it is convenient to select the reference axis in the direction of $\dot{E}_2$, because then $\dot{E}_2$ is determined by its magnitude alone, the direction angle being equal to zero. The generator voltage is expressed by $E_1 (\cos \theta + j \sin \theta)$, where the magnitude of E_1 is given, but the angle θ is unknown. The current lags by the angle ϕ' behind $\dot{E}_2$, and therefore is expressed by the formula $I (\cos \phi' - j \sin \phi')$. Thus eq. (159) becomes

$$E_1 (\cos \theta + j \sin \theta) = E_2 + Iz [\cos (\phi - \phi') + j \sin (\phi - \phi')]. \quad (160)$$

Equating the real and the imaginary parts gives

$$E_1 \cos \theta = E_2 + Iz \cos (\phi - \phi'); \quad . \quad . \quad . \quad (161)$$

$$E_1 \sin \theta = Iz \sin (\phi - \phi'). \quad . \quad . \quad . \quad . \quad . \quad . \quad (162)$$

From eq. (162)

$$\sin \theta = (Iz/E_1) \sin (\phi - \phi'). \quad . \quad . \quad . \quad . \quad . \quad (163)$$

Knowing θ, we find from eq. (161)

$$E_2 = E_1 \cos \theta - Iz \cos (\phi - \phi'). \quad . \quad . \quad . \quad (164)$$

In practice, one is usually required to determine the voltage regulation of the line. According to the definition adopted by the American Institute of Electrical Engineers (*Standards*, see the latest edition),

$$\text{per cent regulation} = 100 (E_2' - E_2)/E_2, \quad . \quad . \quad (165)$$

where E_2' is the value of E_2 at no load. But here $E_2' = E_1$, because the electrostatic capacity of the line is neglected. It is possible to determine the difference $E_1 - E_2$ directly from eq. (161), by substituting for $\cos \theta$ the expression $1 - 2 \sin^2 \frac{1}{2} \theta$. We obtain then

$$\Delta E = E_1 - E_2 = Iz \cos (\phi - \phi') + 2 E_1 \sin^2 \tfrac{1}{2} \theta. \quad (166)$$

Equation (165) becomes

$$\text{per cent regulation} = 100 \, \Delta E/(E_1 - \Delta E). \quad . \quad (167)$$

When it is required to calculate the voltage regulation for several loads, the computations are conveniently arranged in a table of the following form:

Load current I	Line drop Iz	Load power-factor $\cos \phi'$	Angle $\phi-\phi'$	$\sin(\phi-\phi')$	$\cos(\phi-\phi')$	$\sin \theta$	Angle θ	$\sin^2 \frac{1}{2}\theta$	ΔE	Per cent regulation

In practice, the voltage regulation is usually required for a certain load of P_2 watts, so that, strictly speaking, the current I is not known. But, since E_2 is not much different from E_1, it is an easy matter to estimate the current with sufficient accuracy. Or else a curve of voltage regulation is plotted against the load as abscissæ, so that the regulation may be read off at any desired load. It is possible to solve the problem exactly, by using for I its expression $P_2/(E_2 \cos \phi')$ in eqs. (161) and (162). In this case the equations are squared and added, so as to eliminate θ. This gives a biquadratic equation for E_2, from which the receiver voltage can be computed.

This problem can also be solved when the complex quantities are expressed in the orthogonal form, instead of the trigonometric form here used. The student is urged to work out the details, in order to become thoroughly familiar with complex quantities in both forms.

Prob. 1. A vector $72 + j\,53$ must be turned by 25 degrees in the negative direction. What are its new projections? Ans. $87.65 + j\,17.6$.

Prob. 2. A single-phase aluminum line is to be built from a power house, at which a voltage of 11,500 is maintained at a frequency of 50 cycles, to a point 25 km. distant. When a current of 60 amp. at 80 per cent lagging power-factor is delivered at the receiver end, the power loss in the line must not exceed 10 per cent of the useful power. What must be the size of the conductor, and what will be the per cent voltage regulation at this load? The spacing between the wires is to be 61 cm.

Ans. No. 0000 B. & S.; 11.4 per cent.

Prob. 3. Check the answer to the preceding problem graphically.

Prob. 4. Explain the theory of Mershon's diagram found in various electrical handbooks and pocketbooks, and check by means of it the answer to problem 2.

Prob. 5. Show how to determine the voltage regulation of a transmission line when the receiver voltage is given.

Prob. 6. Show how to calculate the receiver voltage E_2 from eq. (159), using the orthogonal projections of the vectors and operators. Discuss the relative advantages and disadvantages of the rectangular and polar coördinates in this case.

34. Vectors and Operators Expressed as Exponential Functions.[1] Expressions (149) to (153) are sometimes written in the exponential form, using the identity

$$\cos\theta + j\sin\theta = \epsilon^{j\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

where ϵ is the base of natural logarithms. This important equation follows from the well-known expansions for $\sin\theta$, $\cos\theta$ and ϵ^{θ}, obtained by Maclaurin's Theorem in calculus; namely,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ . \ . \ . \ ;$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ . \ . \ . \ ;$$

$$\epsilon^{\theta} = 1 + \frac{\theta}{1} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ . \ . \ . \ .$$

The last series, when $j\,\theta$ is substituted for θ, becomes

$$\epsilon^{j\theta} = 1 + j\,\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ . \ . \ . \ .$$

Substituting these values into eq. (168), it is found to be an identity. Thus, we have

$$\dot{E} = E(\cos\theta + j\sin\theta) = E\epsilon^{j\theta}. \quad . \quad . \quad . \quad . \quad (169)$$

Similarly, the impedance operator becomes

$$Z = z(\cos\phi + j\sin\phi) = z\epsilon^{j\phi}, \quad . \quad . \quad . \quad . \quad (170)$$

and the admittance operator

$$Y = y(\cos\phi - j\sin\phi) = y\epsilon^{-j\phi}. \quad . \quad . \quad . \quad (171)$$

If, for instance, a current is given as $\dot{I} = I\epsilon^{j\theta}$ and if it flows through an impedance $Z = z\epsilon^{j\phi}$, the required voltage is found by multiplying these two expressions, or

$$\dot{E} = \dot{I}Z = Iz\epsilon^{j(\theta+\phi)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

[1] This article may be omitted if desired, without impairing the continuity of the treatment in the rest of the book.

This shows that the absolute value of the voltage is Iz and that it leads the current by the angle ϕ. Equation (172) corresponds to eq. (157) in trigonometric notation. The projections of the current vector are $I \cos \theta$ and $I \sin \theta$; while the projections of the voltage vector are $Iz \cos (\theta + \phi)$ and $Iz \sin (\theta + \phi)$. Thus, it is always possible to change from the exponential form to the trigonometric form, and finally to the orthogonal projections, or vice versa. The exponential form is more concise, and possesses marked advantages in the solution of some advanced problems relating to alternating currents and oscillations.[1] However, for the simple problems treated in this book, the plain algebraic notation $a + ja'$ and the trigonometric notation $A (\cos \alpha + j \sin \alpha)$ are amply sufficient. It is deemed advisable to explain the exponential notation here in order to enable the student to read books and magazine articles in which it is employed.

[1] See for instance J. J. Thomson, *Recent Researches in Electricity and Magnetism.*

CHAPTER X

POLYPHASE SYSTEMS

35. Two-phase System. The student knows from his elementary work that the induction motor operates on the principle of the revolving magnetic field, and that such a field is produced by a combination of two or more alternating currents differing in phase. An electric circuit upon which are impressed two or more waves of e.m.f. having definite phase displacements is called a polyphase system. A large majority of the alternating-current circuits used in practice in the generation and transmission of electrical energy are polyphase systems; it is therefore essential that the student become familiar with the current and voltage relations in such circuits.

Theoretically, the simplest polyphase system is a *four-wire two-phase system* (Fig. 30), although it is not the most econom-

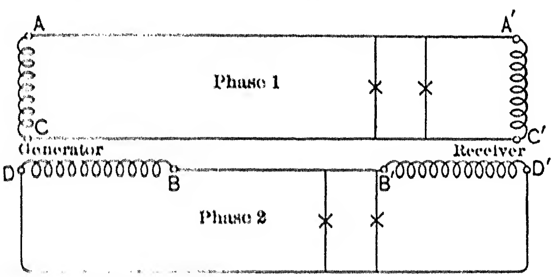

FIG. 30. A four-wire two-phase system with two independent circuits.

ical one in practice. The two generator windings are independent, and are relatively displaced by ninety electrical degrees. The two alternating voltages induced in these windings are therefore displaced in time phase by a quarter of a cycle. Each phase may be used separately, for instance for lighting, or both phases may be combined in the windings of a synchronous or induction motor, for the production of a revolving magnetic field. Each phase can be treated separately, as if it belonged to an independent single-phase system.

Some economy in line conductors and insulators is achieved by combining two conductors belonging to different phases into one return conductor (Fig. 31). Such a system is called a *three-wire*

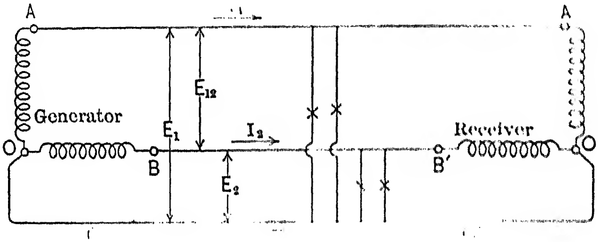

FIG. 31. A three-wire two-phase system.

two-phase system. The current and voltage relations for a balanced load and a lagging current are shown in Fig. 32. The vectors E_1 and E_2 represent the voltages induced in the two generator or transformer windings from the point O to the points A and B respectively; in other words, they are the voltages between each phase wire and the return wire. The vector E_{12}* is the geometric *difference* of the two, and represents the voltage between the two phase wires. That E_{12} is the difference and not the sum of E_1 and E_2 is proved by the following reasoning: Let the wire $OGG'O'$ be permanently

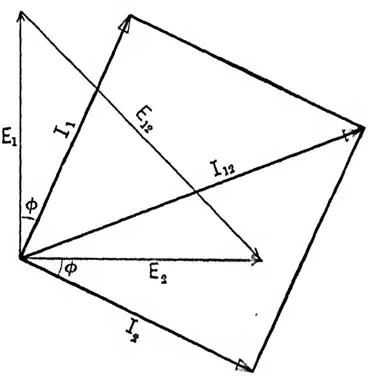

FIG. 32. A vector diagram of currents and voltages for the two-phase system shown in Fig. 31.

grounded, so that its potential is zero. Let the potential of the wire AA' at a certain instant be for example 100 volts above the ground, and that of the wire BB' 60 volts above the ground. Then the difference of potential, or the voltage between AA' and BB', is 40 volts. The same reasoning applies to every instant, so that the vector of the voltage between AA' and BB' is the geometric difference between E_1 and E_2, which are the vectors of the voltages between the points A and O, and B and O re-

* Pronounced E — one — two, and not E sub twelve.

spectively. That is, the voltage between A and B is represented in phase and magnitude by the vector connecting the ends of the vectors E_1 and E_2. Numerically,

$$E_{12} = E_1 \sqrt{2} = E_2\sqrt{2}. \quad . \quad . \quad . \quad . \quad . \quad (173)$$

The currents in the conductors AA' and BB' are represented by the vectors I_1 and I_2, lagging by an angle ϕ with respect to the corresponding voltages. The current in the return conductor is the geometric *sum* of the two phase currents, and is represented by the diagonal vector I_{12}. It will thus be seen that the common return current is $\sqrt{2}$ times as large as each component current, or

$$I_{12} = I_1 \sqrt{2} = I_2\sqrt{2}. \quad . \quad . \quad . \quad . \quad . \quad (174)$$

If it is desired to have the same current density in each of the three conductors, the cross-section of the return wire must be $\sqrt{2}$ times that of each of the other two wires.

The two phases in Fig. 30 are sometimes electrically interconnected at their middle points, as shown in Fig. 33 at the left.

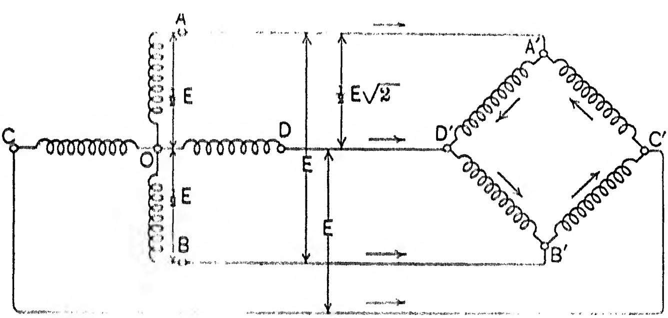

FIG. 33. A star-connected quarter-phase system to the left, a mesh-connected system to the right.

This is done in order to fix the difference of potential between the two phases. If the voltage between A and B is E, then the voltage between the common point O and each wire is $\frac{1}{2} E$, and the voltage between the two phases is equal to $\frac{1}{2} E \sqrt{2} = E/\sqrt{2}$. For example, the voltage E_{da} between D and A equals $E_a - E_d$. These relations are shown vectorially in Fig. 34. This circuit is sometimes called the *star*-connected quarter-phase system.

The four windings of a generator or motor are sometimes connected in *mesh*, as indicated in Fig. 33 to the right. With the

star connection, the star voltages OA, OB, etc., are induced directly, while the mesh voltages AD, DB, etc., are established by the combination of the star voltages. With the mesh connection of the windings, however, the mesh or line voltages are induced directly. The mesh and star voltages are shown in Fig. 34. Electrically the two arrangements are equivalent, provided that the proper numbers of turns are used in the windings.

The line and mesh currents are indicated in Fig. 34. The line currents and those in the star-connected windings are represented

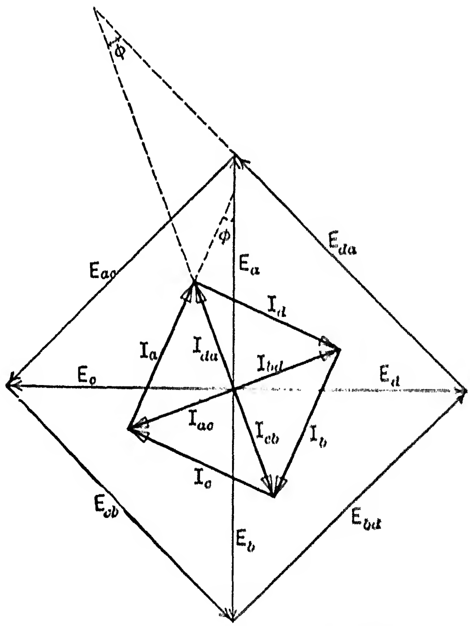

Fig. 34. A vector diagram of currents and voltages in the quarter-phase system shown in Fig. 33.

by the sides of the square, each current lagging by the angle ϕ with respect to the corresponding star voltage; the angle of lag depends upon the character of the load. In Figs. 33 and 34 the voltages are taken in the cyclic order AC, CB, BD, DA; hence, it is natural to take the positive direction of the current in the same way. With the arrows in Fig. 33 showing the positive directions of the currents, each line current is the difference between two adjacent mesh currents. Hence, in the vector diagram the mesh

currents are represented by the radii from the center to the vertices of the current square. It will be seen that the angle between the mesh currents and the mesh voltages is also equal to ϕ. While the mesh voltages are $\sqrt{2}$ times as large as the star voltages, the mesh currents are $1/\sqrt{2}$ times the star currents. This condition is necessary in order to have the same power per phase in the mesh and star-connected systems.

36. Three-phase Y-connected System. This system is shown in Fig. 35; the current and voltage relations are repre-

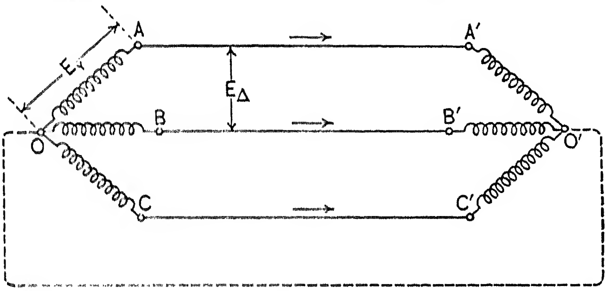

FIG. 35. A three-phase Y- or star-connected system.

sented in Fig. 36. OA, OB, and OC represent three generator windings; $O'A'$, $O'B'$, and $O'C'$ are the windings of a receiving apparatus—for instance, an induction motor. The three generator windings are placed on the armature core at angles of 120 electrical degrees with respect to each other, so that the alternating voltages induced in these windings are displaced in phase by one-third of a cycle, the positive direction in the windings being outward. They are represented by the vectors E_a, E_b, and E_c in Fig. 36. The motor windings are similarly displaced, so that the whole system is symmetrical with respect to the three phases. The currents lag behind the voltages by an angle ϕ depending upon the relative amounts of resistance, reactance, and counter-e.m.f. in the circuit.

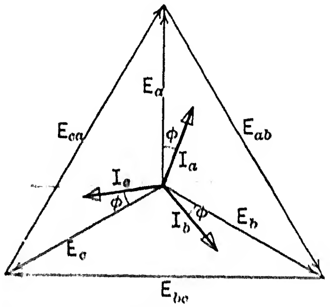

FIG. 36. The line and star voltages, and the line currents in the Y-connected system shown in Fig. 35.

The diagram of connections shown in Fig. 35 is also called the *star* connection, and the points O and O' are called the neutral points of the system. The voltages between the line conductors and the neutral points are called the star or phase voltages, as distinguished from the line voltages, or voltages between any two line conductors. A line voltage, for instance between the points A and B, is equal to the geometric difference between the voltages OA and OB, as has been shown above in the case of a two-phase line. Consequently, the line voltages are represented in Fig. 36 by the three vectors E_{ab}, E_{bc}, and E_{ca}, which connect the ends of the vectors of the phase voltages. It will be seen that the line voltages are $\sqrt{3}$ times as large as the phase voltages.

When the three phases are perfectly balanced and the currents are nearly sinusoidal, the two neutral points O and O' may be connected by a wire, as shown by the dotted line, or grounded, and very little current will flow through this connection. The reason is that the algebraic sum of the three currents flowing towards or from the neutral points is equal to zero at all instants, because

$$\sin u + \sin (u + \tfrac{2}{3}\pi) + \sin (u - \tfrac{2}{3}\pi) = 0. \qquad (175)$$

This identity is easily proved by expanding the left-hand member, using the expression for the sine of the sum of two angles. It will also be seen from Fig. 36 that the geometric sum of the three current vectors is equal to zero, because these vectors when added form a closed triangle. In practice, there are transmission lines on which one or both neutrals are grounded, although in some installations both neutrals are insulated from the ground. To prevent large currents with unbalanced loads or during short-circuits, the neutrals are often grounded through protective resistances. The question of grounded *vs.* ungrounded neutrals is still in a somewhat controversial stage.

The power developed in the generator windings and available at the generator terminals is $3\, I_Y E_Y \cos \phi$, where E_Y is the phase voltage. Since the line voltage $E_\Delta = E_Y \sqrt{3}$, we have

$$P = 3\, I_Y E_Y \cos \phi = I_Y E_\Delta \sqrt{3} \cos \phi. \quad . \quad . \quad (176)$$

In practical calculations of three-phase transmission lines and electrical machinery, only one phase is considered; that is, the three-phase circuit is reduced to an equivalent single-phase circuit. Let it be required, for example, to calculate the cross-sec-

tion and per cent voltage regulation of a three-phase 66,000-volt line, to transmit 50,000 kw. at 80 per cent power-factor, and at a loss of 10 per cent of the useful power; the spacing to be 1.8 m. First of all we find that the voltage between each wire and the neutral is $66,000/\sqrt{3} = 38,100$ volts, and that the power per phase is $50,000/3 = 16,700$ kw. Hence, the problem is reduced to the following one: Determine the cross-section and per cent voltage regulation of a single-phase 38,100-volt line, having a spacing of 1.8 m., the i^2r loss in one conductor being 1670 kw., and the resistance of the return conductor being negligible. The solution of this problem is given in Art. 33 above. A drop of say 5 per cent in the phase or star voltage means also a drop of 5 per cent in the line voltage, because of the fixed ratio $1/\sqrt{3}$ between the two.

Prob. 1. Assuming the reference axis in Fig. 36 to be horizontal, the line voltage equal to 44 kv., the current per phase 73 amp., and the angle ϕ equal to 15 degrees, write down the complex expressions for all the currents and voltages.

Ans. $E_b = 22 - j\,12.7$ kv.; $E_{ab} = 22 - j\,38.1$ kv.; $I_a = 18.9 + j\,70.5$ amp.

Prob. 2. A three-phase 60-cycle line is 16 km. long; the spacing between the wires is symmetrical and is equal to 61 cm., the conductors consisting of copper wire of 14 mm. diameter. It is required to maintain a voltage of 6700 between the conductors at the receiver end of the line. What is the generator voltage when the load is equal to 1000 kw. at unity power-factor? Ans. 7040.

Prob. 3. Show that a three-phase transmission line may be treated as a single-phase line which transmits one-half the power at the same voltage. The three-phase line requires three conductors of the same size as the single-phase line, with the same spacing (25 per cent saving in material). Per cent power loss is the same in both.

Prob. 4. When the phase currents have higher harmonics, multiple of three, show that equalization currents must flow through the neutral connection, even though the phases are perfectly balanced. What happens when the neutrals are insulated from each other?

Prob. 5. Show that the line voltage cannot have the third, the ninth, the fifteenth, etc., harmonics, even if these harmonics are present in the phase voltages.

37. Three-phase Delta-connected System. This method of three-phase connection is shown in Fig. 37, one end of the line being connected, for instance, to an alternator, the other end to a motor or to three transformers. Fig. 38 represents the current

and voltage relations. The currents in the windings are different from those in the line. With the positive directions of the

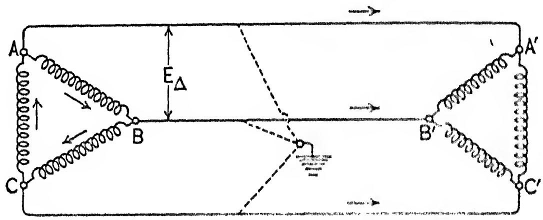

FIG. 37. A three-phase delta- or mesh-connected system.

currents indicated in Fig. 37, each line current is equal to the difference between the two adjacent currents in the "delta."

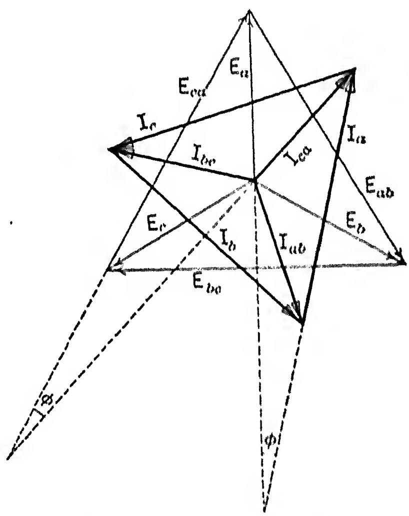

FIG. 38. The voltages and currents in the delta-connected system shown in Fig. 37.

Hence, in the vector diagram the line or "star" currents are represented by a triangle, and the delta currents by the rays from the center to the vertices of the triangle. It will be seen that the delta currents are equal to $1/\sqrt{3}$ of the line currents, and are displaced in phase by 30 degrees with respect to them. This also follows from the identity

$$I \sin u - I \sin (u + 120°) \equiv \sqrt{3}\, I \sin (u - 30°). \quad . \quad (177)$$

While there are no neutral points with a delta system, one or more of them may be artificially created by connecting three resistances or reactances in star as shown in Fig. 37 by dotted lines. The Y-voltages between this neutral and the line conductors are shown in Fig. 38 by the vectors E_a, E_b and E_c. The delta voltages are $\sqrt{3}$ times as large as the star voltages, while the star or line currents bear the same ratio to the delta currents. This is necessary in view of the power relation

$$P = 3\, E_{\Delta} I_{\Delta} \cos\phi = 3\, E_{Y} I_{Y} \cos\phi. \quad . \quad . \quad . \quad (178)$$

In design and performance calculations one phase only is considered, the three phases being identical when the load is balanced. As far as the line is concerned, the delta-connected generator and load may be replaced by equivalent star-connected windings to give the same line currents and voltages. Then the line is designed and its performance calculated the same as in the preceding article. As a matter of fact, for line calculations it is only necessary to know the power, the voltage, and the power-factor of the load. The fact that the generator or the load is delta- or Y-connected has no bearing upon the line performance with a balanced load.

Prob. 1. A 2000-kw. 6600-volt induction motor is fed from a 66,000-volt three-phase line through three step-down transformers, the high-tension windings of which are connected in Y, the low-tension windings in delta. What are the currents in these windings when the motor is carrying a 25 per cent overload? It is estimated that at this overload the power-factor is 90 per cent and the efficiency 92 per cent. The magnetizing current of the transformers is negligible.

Ans. 26.4 and 153 amp.

Prob. 2. Show that, while the instantaneous electrical output of a single-phase alternator varies at double the frequency of the current, the output of a polyphase machine is practically constant as long as the load remains constant. Show that the same is true for motors.

Note: For the electrical relations in two- and three-phase systems with unbalanced loads, and also for the theory of the V and T connections, see the author's *Experimental Electrical Engineering*, Vol. 2, Chapter 25. A more exhaustive treatment will also be found in his investigation entitled *Ueber mehrphasige Stromsysteme bei ungleichmässiger Belastung* (published by Enke, 1900). See also the chapters on polyphase systems in Dr. Steinmetz's *Alternating-current Phenomena.*

CHAPTER XI

VOLTAGE REGULATION OF THE TRANSFORMER

38. Imperfections in a Transformer Replaced by Equivalent Resistances and Reactances. The reader is familiar in general with the construction and operation of the constant-potential transformer (Fig. 39). It consists of an iron core upon which two windings are placed as closely as possible to each other. When one winding is connected to a constant-potential alternating-current source of power, an alternating magnetic flux is excited in the iron core and an alternating voltage is induced in the other winding. If this latter winding is connected to an electrical load, an alternating current flows through it, and causes a corresponding flow of current through the first winding, in order that power may be transmitted from the primary into the secondary circuit.

The constant-potential transformer is one of the most perfect pieces of electrical apparatus, in that its efficiency (in medium and large sizes) is nearly one hundred per cent, and its voltage regulation with varying load is quite close. On the other hand, the requirements for voltage regulation are quite exacting, there being no provision in the apparatus itself for adjusting the voltage, like the field rheostat in a generator. Therefore, the pre-determination of the voltage regulation of a transformer is of considerable practical importance.

Numerically, the regulation of a transformer is expressed in a manner similar to that given in Art. 33 above for the transmission line. Let, for example, the rated secondary voltage of a ten-to-one transformer be 220 volts, and let us suppose that a primary e.m.f. of 2280 volts is necessary in order to have the rated secondary voltage at the rated load. Let now the secondary circuit be opened; the secondary voltage will rise to practically 228 volts, provided that the primary voltage is kept constant. Then, by definition, the regulation of the transformer at this load is $100 (228 - 220)/220 = 3.64$ per cent. Let, in general,

the secondary terminal voltage at a certain load be E_2, that at no load E_{02}. Then, by definition,

$$\text{per cent regulation} = 100\,(E_{02} - E_2)/E_2. \quad . \quad . \quad (179)$$

The difference between the no-load voltage and that at full load is due to slight imperfections in the transformer itself. Therefore, in order to be able to calculate the voltage regulation at a given load, it is necessary to learn the nature of these imperfections; for purposes of computation, it is convenient to replace these imperfections by certain resistances and reactances, as shown in Fig. 39.

In an ideal transformer the ratio of the primary to the secondary voltage is equal to the ratio of the numbers of turns in the corresponding windings. The same relation is very nearly true in any good transformer at no load. This follows from the fact that the two windings are linked with the same magnetic flux, and hence the voltage induced *per turn* is the same in both. Hence, denoting the primary and secondary *induced* voltages by E_{i1} and E_{i2}, and the corresponding numbers of turns in series by n_1 and n_2, we have

$$E_{i1}/E_{i2} = n_1/n_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

Furthermore, in an ideal transformer

$$I_1 n_1 = I_2 n_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

that is, the currents are inversely as the numbers of turns, or the primary ampere-turns are equal and opposite to the secondary ampere-turns. This is because an ideal transformer is supposed to have no reluctance in its magnetic circuit, so that no ampere-turns are required to maintain a magnetic flux in it. Consequently, any secondary current required by the load automatically draws a compensating primary current from the source of power, of such value that eq. (181) is satisfied. In a real transformer the primary ampere-turns are slightly different from the secondary ampere-turns, and the difference between the two is just sufficient to maintain the alternating flux through the reluctance of the core, and to supply the core loss. Multiplying eqs. (180) and (181) term by term, and canceling n_1 and n_2, we find that

$$E_{i1} I_1 = E_{i2} I_2.$$

This simply means that an ideal transformer transmits power from the primary into the secondary circuit without loss.

(a) *The Ohmic Drop.* One of the causes of the internal voltage drop in a transformer is the ohmic resistance of its windings. Because of the resistance of the primary winding, the primary terminal voltage E_1 (Fig. 39) is slightly larger than the induced

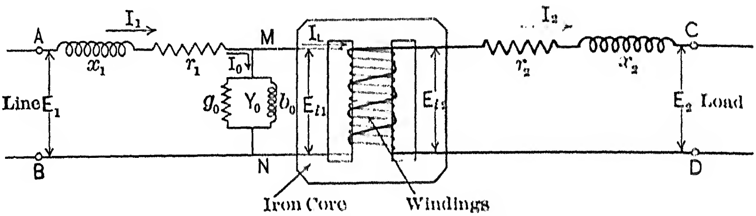

FIG. 39. Imperfections in a transformer represented by resistances and reactances.

counter-e.m.f. E_{i1} which balances it. The secondary resistance causes a voltage drop, so that the secondary terminal voltage E_2 is smaller than the secondary induced e.m.f. E_{i2}. Thus, the effect of the internal resistances upon the terminal voltages is such as to make the ratio E_2/E_1 smaller than the ratio n_2/n_1. The windings themselves may be thought of as devoid of resistance, but corresponding resistances r_1 and r_2 may be placed outside the transformer, as shown in Fig. 39.[1]

(b) *The Reactive Drop.* Another imperfection or cause of internal voltage drop is the so-called leakage reactance of the windings. The total magnetic flux in a loaded transformer may be considered as consisting of three components; *viz.*, the useful flux linking with both the primary and the secondary windings, the primary leakage flux linking with the primary winding only, and the secondary leakage flux linked with the secondary winding only. In an ideal transformer the two last-named fluxes are absent because the two windings are supposed to be perfectly interwoven, so as to leave no room for the leakage flux. The primary leakage flux, being produced by the primary current, is in phase with it, and induces an e.m.f. in lagging quadrature with

[1] The equivalent resistances r_1 and r_2 must replace not only the true ohmic resistances of the windings, but should also account for the eddy-current loss in the conductors. In low-tension windings made of heavy conductors, this latter loss may be at least as great as the theoretical i^2r loss. In new transformers the eddy-current loss can only be estimated; in actually built transformers it is calculated from the wattmeter reading on short circuit.

this current. This e.m.f. must be balanced by part of the applied primary voltage, so that either this voltage or the induced e.m.f. E_{i1} must be different from that in an ideal transformer. The effect of the secondary leakage reactance is similar, in that it absorbs part of the secondary induced voltage E_{i2}, and makes the secondary terminal voltage E_2 different from that in an ideal transformer. It is shown below that, with a load of lagging power-factor, the reactive drop in both windings lowers the secondary terminal voltage. With a leading secondary current, the reactive drop is in such a phase position as to raise the secondary voltage. For purposes of computation, the transformer windings are assumed to produce no magnetic leakage fluxes, but imaginary reactance coils are connected in series with the windings (Fig. 39). The reactances x_1 and x_2 of these coils are such as to cause the same reactive voltage drop as that due to the actual leakage fluxes in the transformer.[1]

(c) *The Exciting Admittance.* Having thus made the windings of the transformer perfect by placing their impedances outside, we still have the problem of making the magnetic circuit ideal also. As stated before, the primary and secondary ampere-turns are not quite equal, because of a certain number of ampere-turns necessary to magnetize the iron. This means that a current must flow through the primary winding even when the secondary circuit is open. This current is called the no-load or magnetizing current of the transformer. Its amount depends upon the reluctance of the magnetic circuit and upon the core loss (hysteresis and eddy currents). For purposes of computation the iron core may be assumed to be of zero reluctance, and to have no core loss; but we may imagine a fictitious or equivalent susceptance b_0 and a conductance g_0 (Fig. 39) connected across the primary winding to draw a current equal in phase and magnitude to the exciting current of the transformer. Both g_0 and b_0 are shown connected across the induced voltage E_{i1}, because both the magnetizing current and the core loss depend upon the value of the flux and consequently upon the value of E_{i1}, which is proportional to the flux. Let the calculated or measured core loss be equal to P_0 watts; then g_0 is determined from the equation

$$P_0 = E_{i1}^2 g_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

[1] For further details in regard to the leakage reactance of transformers, see the author's *Magnetic Circuit*, Art. 64.

The pure magnetizing current, without the core-loss component, is in phase with the flux which it produces, and therefore is in quadrature with the induced voltage E_{i1}. For this reason, it is represented as flowing through a pure susceptance. Knowing the pure magnetizing current I_0', the susceptance b_0 is determined from the equation[1]

$$b_0 = I_0'/E_{i1}. \quad \ldots \ldots \ldots \quad (183)$$

Neither the core-loss component nor the pure magnetizing current are proportional to the flux or to the voltage E_{i1}, so that strictly speaking both b_0 and g_0 are functions of the counter-e.m.f. E_{i1}. However, in practice, E_{i1} varies so little with the load that it is admissible to assume g_0 and b_0 to be constant quantities. Moreover, the influence of the magnetizing current upon the voltage regulation is negligible in most cases. The magnetizing current is mentioned here only for the sake of completeness, so as to make the transformer core absolutely perfect. We shall see in the next chapter that b_0 and g_0 are of considerable importance in the performance of the induction motor.

Thus, by the foregoing reasoning, both the core and the windings of the transformer are made ideal, and all the imperfections are replaced by external resistances and reactances. Having done this, the performance of a transformer can be readily treated either graphically or analytically, as explained below.

Prob. 1. Draw a diagram similar to Fig. 39 for a transformer with several secondary windings supplying independent load circuits.

Prob. 2. Draw a diagram similar to Fig. 39 for an auto-transformer.

39. The Vector Diagram of a Transformer. Having reduced the transformer to an equivalent electric circuit (with a perfect magnetic link), the current and voltage relations at a certain load may be represented by a vector diagram (Fig. 40). In order to make the relations clearer, the voltage drop and the losses are greatly exaggerated. For this reason, the graphical treatment is more suitable for purposes of explanation than for numerical computations. For actual calculations the analytical method given in the next article is preferable.

[1] The calculation of the core loss and of the magnetizing current belongs properly to the theory of magnetic phenomena, and is treated in detail in the author's *Magnetic Circuit*, Arts. 19, 33, and 34. Here the values of P_0 and I_0' are supposed to be known.

Let the secondary terminal voltage and the load be given, so that the vectors E_2 and I_2 can be drawn in magnitude and relative phase position. The secondary induced e.m.f. E_{i2} is found

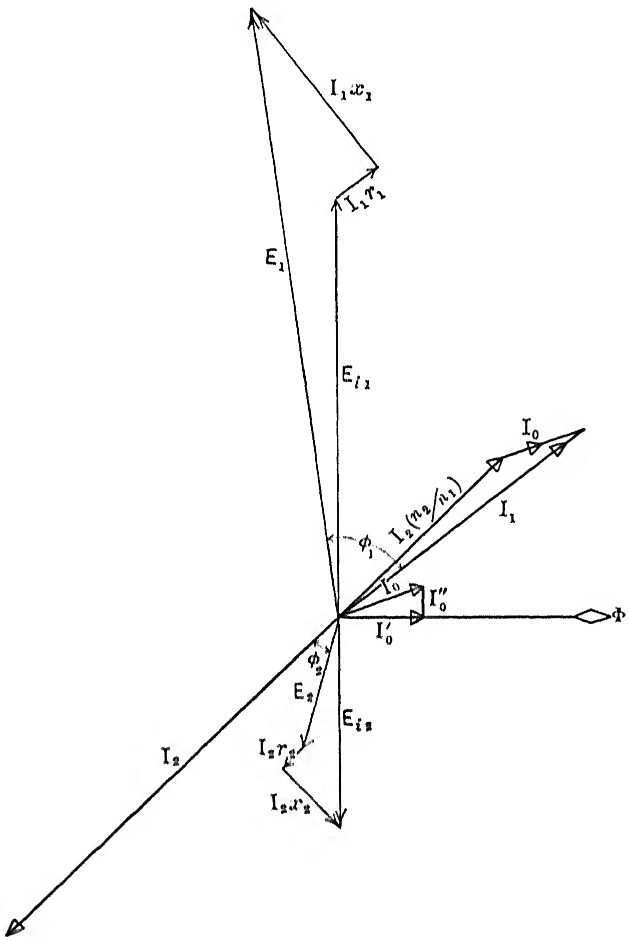

FIG. 40. The vector diagram of a transformer.

by adding to E_2 the ohmic drop I_2r_2 in phase with I_2, and the reactive drop I_2x_2 in leading quadrature with I_2.

The primary induced voltage E_{i1} is in phase with E_{i2}, because both are induced by the same magnetic flux. The magnitudes of the two voltages are as the respective numbers of turns; see

eq. (180). The vector marked E_{i1} in Fig. 40 is in reality equal and opposite to E_{i1}, and represents the part of the primary terminal voltage that balances E_{i1}. Without the primary drop, the total applied primary voltage would be equal to E_{i1}. But, on account of the primary drop in the transformer, the applied voltage E_1 is obtained by adding to E_{i1} the resistance drop I_1r_1 in phase with the primary current I_1 and the reactive drop I_1x_1 in leading quadrature with I_1.

In order to be able to construct the vectors I_1r_1 and I_1x_1 it is necessary to know the vector of the primary current I_1 in magnitude and phase position. In an ideal transformer, the primary current is in exact phase opposition to the secondary current, and the ratio of the two currents is inversely as the ratio of the respective numbers of turns; see eq. (181). In the actual transformer, the primary current, in addition to this component $I_2\,(n_2/n_1)$ "transmitted into the secondary," has a magnetizing component I_0, which serves to maintain the alternating flux in the core, and which is not transmitted into the secondary circuit. The total primary current is the geometric sum of the two components, and can be constructed if the magnetizing current I_0 is known.

The magnetizing current itself consists of two components, as explained in the preceding article, under (c). One component, I_0', is in phase with the useful magnetic flux Φ, and would be the only magnetizing component if the iron had no hysteresis and no eddy currents. This component is in quadrature with the induced voltage E_{i1}, and is, with respect to it, the reactive component of the magnetizing current. The other component, I_0'', in phase with E_{i1}, represents a loss of power, and is therefore called the energy or loss component of the magnetizing current. Knowing I_0' and I_0'', the vector I_0 is easily obtained.

The vector of flux, Φ, is drawn in leading quadrature with the induced e.m.f. E_{i2}, in accordance with Faraday's law of induction. If the flux varies according to the law

$$\Phi_t = \Phi_m \sin 2\,\pi f t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

the induced e.m.f. varies according to the law

$$e_{i2} = -n_2\, d\Phi_t/dt = -2\,\pi f \Phi_m n_2 \cos 2\,\pi f t, \quad . \quad . \quad (185)$$

the second sine-wave lagging by 90 degrees behind the first.

It is assumed in the construction of Fig. 40 that the primary and secondary inductive drops can be calculated separately. Such is, however, not the case with our present state of knowledge; both theory and experiment enable us to determine only the total reactive drop, including primary and secondary. Therefore, when it is desired to use the vector diagram for actual computations, it is customary to ascribe one half of the Ix drop to the primary and the other half to the secondary circuit.

Usually, the magnetizing component of the primary current can be neglected; then it does not make any difference how the inductive drop is distributed. It will be shown in the next article that in such case the voltage regulation depends only upon the total impedance drop, either calculated or determined from a short-circuit test. When the internal voltage drop is given in per cent, it is understood to refer to the no-load voltage of each particular circuit. For instance, if the reactive voltage drop in a 20/1-kv. transformer is said to be 5 per cent, this means that the secondary drop is 2.5 per cent of 1000 volts, or is equal to 25 volts, and that the primary drop is 2.5 per cent of 20,000 volts, or is equal to 500 volts.

Prob. 1. What is the regulation of a 600-kva., 2200/220-volt, 25-cycle transformer at the rated current and at 80 per cent power-factor (lagging)? The total reactive drop is 10 per cent, the primary ohmic drop is 2.2 per cent, and the secondary ohmic drop 2.8 per cent. The magnetizing current may be neglected.[1] Ans. 10.1 per cent.

Prob. 2. Determine the per cent voltage regulation of the transformer specified in the preceding problem at the rated load and at 80 per cent power-factor, *leading*.

Ans. −1.4 per cent. The negative sign indicates a rise in secondary voltage, instead of a drop.

Prob. 3. Correct the vector diagram of problem 1 for the magnetizing current, knowing that the core loss amounts to 20 kw., and that 8500 effective ampere-turns are necessary to maintain the flux, without the iron loss. The number of turns in the secondary winding is 64.

Prob. 4. Adapt the diagram shown in Fig. 40 to an auto-transformer.

40. Analytical Determination of Voltage Regulation.—Approximate Solution. As explained above, it is preferable to calculate the voltage regulation of a transformer analytically,

[1] An excessive internal drop is selected purposely to enable the student to construct an accurate vector diagram to a convenient scale. The losses and the magnetizing current in problem 3 also are too high for a standard transformer.

because the vectors of voltage drop are very small as compared with those of the primary and secondary voltages. The relations shown in Figs. 39 and 40 are expressed analytically by the equations

$$\dot{E}_2 = \dot{E}_{i2} - \dot{I}_2 Z_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (186)$$

and

$$\dot{E}_1 = \dot{E}_{i1} + \dot{I}_1 Z_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (187)$$

Since our purpose is to find the relation between $\dot{E}_1$ and $\dot{E}_2$, it is necessary to eliminate from these equations $\dot{E}_{i1}$ and $\dot{E}_{i2}$. The relation between E_{i1} and E_{i2} is given by eq. (180); therefore, we multiply eq. (186) by n_1/n_2 and subtract it from eq. (187). The result is

$$\dot{E}_1 - (n_1/n_2)\dot{E}_2 = \dot{I}_1 Z_1 + (n_1/n_2)\dot{I}_2 Z_2. \quad . \quad . \quad . \qquad (188)$$

The correct relation between $\dot{I}_1$ and $\dot{I}_2$ is (Fig. 39)

$$\dot{I}_1 = \dot{I}_2 (n_2/n_1) + \dot{I}_0 = \dot{I}_L + \dot{I}_0, \quad . \quad . \quad . \quad . \qquad (189)$$

where

$$\dot{I}_L = \dot{I}_2 (n_2/n_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (190)$$

is the *primary load current*, or that part of the primary current which is transmitted into the secondary circuit. In a great majority of practical cases the magnetizing current is only a few per cent of the total primary current at the rated load. The voltage drop in the primary winding is also but a few per cent of the line voltage E_1. For these reasons, it is permissible in Fig. 39 to transfer the exciting admittance Y_0 from the place MN to the primary terminals AB. The voltage drop in the transformer is then caused only by the load current, so that for the purpose of calculating regulation we may use the *approximate* relation

$$\dot{I}_1 = \dot{I}_L = \dot{I}_2 (n_2/n_1). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (191)$$

Substituting for $\dot{I}_1$ and $\dot{I}_2$ their values from eq. (191) in terms of $\dot{I}_L$, we finally obtain

$$\dot{E}_1 - \dot{E}_L = \dot{I}_L[Z_1 + (n_1/n_2)^2 Z_2]. \quad . \quad . \quad . \quad . \quad . \qquad (192)$$

In this equation, the quantity

$$E_L = (n_1/n_2)E_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (193)$$

is called the *primary load voltage*, or the secondary terminal voltage reduced to the primary circuit. The expression $(n_1/n_2)^2 Z_2$ is called the secondary impedance reduced to, or transferred into, the primary circuit. The quantity

$$Z = Z_1 + (n_1/n_2)^2 Z_2 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (194)$$

is called the total or equivalent impedance of the transformer reduced to the primary circuit.

Using in eq. (192) the abbreviated notation introduced in eq. (194), we get

$$\dot{E}_1 - \dot{E}_L = \dot{I}_L Z. \quad . \quad . \quad . \quad . \quad . \quad . \quad (195)$$

Equation (195) corresponds to the simplified equivalent diagram of the transformer shown in Fig. 41. This diagram differs from Fig. 39 in two respects: (1) The magnetic link is omitted, the primary circuit being connected directly to the modified secondary circuit; (2) the exciting admittance is connected across the primary terminal voltage instead of across the induced voltage. The latter change makes the equivalent diagram only approximately correct, but simplifies computations greatly.

Equation (195) is identical in form with eq. (159), Art. 33, for the voltage drop in a transmission line; both are solved, and the per cent voltage drop determined, in the same way. In fact, without the exciting admittance Y_0, the equivalent diagram shown in Fig. 41 reduces the performance of a transformer to that of a transmission line.

Expression (194) for the equivalent impedance shows that resistances and reactances can be transferred from the secondary

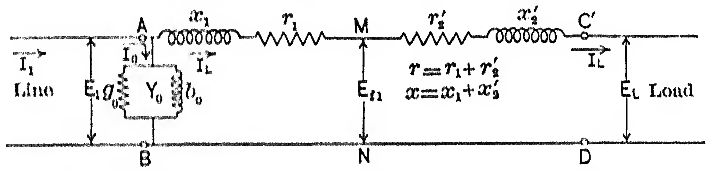

FIG. 41. The approximately equivalent diagram of a transformer or an induction motor.

circuit into the primary, and vice versa, by multiplying them by the square of the ratio of the numbers of turns. For instance, in a 10,000/1000-volt transformer, a 1-ohm resistance in the low-tension circuit causes the same per cent voltage drop as a 100-ohm resistance in the high-tension circuit. This is easily verified as follows: Let the current in the low-tension circuit be 20 amp.; then in the high-tension circuit the current will be 2 amp. The drop in the 1-ohm resistance is 20 volts, or 2 per cent of the secondary voltage. The drop in the 100-ohm resistance is 200 volts, which is 2 per cent of the primary voltage. In other words, the same reduction in the load voltage will be produced by using

either a resistance of one ohm in the secondary circuit or 100 ohms in the primary circuit.

The secondary resistance r_2 transferred into the primary circuit is denoted in Fig. 41 by r_2', where

$$r_2' = r_2 (n_1/n_2)^2. \qquad (196)$$

Correspondingly

$$x_2' = x_2 (n_1/n_2)^2. \qquad (196a)$$

The equivalent impedance Z consists of the quadrature sum of the equivalent resistance

$$r = r_1 + r_2' = r_1 + (n_1/n_2)^2 r_2, \qquad (197)$$

and the equivalent reactance

$$x = x_1 + x_2' = x_1 + (n_1/n_2)^2 x_2 \qquad (197a)$$

The equivalent resistance is easily calculated, knowing the resistances of the two windings and the voltage ratio of the transformer. Or else it is calculated directly from the i^2r loss measured by a wattmeter in a short circuit test. The equivalent reactance is calculated from the terminal voltage in the short-circuit test, making a proper allowance for the known resistance drop. To illustrate, when the secondary circuit is short-circuited, eq. (195) becomes

$$\dot{E}_1 = \dot{I}_L Z. \qquad (198)$$

E_1 and I_L are measured directly, so that Z can be calculated. Knowing the equivalent resistance $r = P/I_L^2$, the reactance is calculated from the expression $x = \sqrt{z^2 - r^2}$. For a new transformer, the total equivalent leakage inductance is estimated with sufficient accuracy by means of various semi-empirical formulæ;[1] or else the total impedance drop $I_L Z$ is taken as a certain percentage of the rated voltage, from previous experience with similar transformers.

Prob. 1. Check analytically the answers to problems 1 and 2 in the preceding article.

Prob. 2. The high-tension winding of a 2000-kva., 33/11-kv. transformer was short-circuited, and the voltage on the low-tension side adjusted so as to circulate the rated current through the windings. The instrument readings were 470 volts and 30 kw. Calculate the per cent ohmic and reactive drops in the transformer. Ans. 1.5 and 4 per cent.

Prob. 3. Deduce a formula similar to (192), but referring to the secondary circuit.

[1] See for instance the author's *Magnetic Circuit*, Art. 64.

Prob. 4. Show how the voltage regulation of a transformer can be estimated, using Mershon's diagram given in various electrical handbooks and pocketbooks.

Prob. 5. The primary voltage of a given transformer is kept constant at a known value. Determine the percentage internal drop $(E_1 - E_L)/E_1$ for a given impedance of the load.[1] Solution: Let the load impedance, reduced to the primary circuit, be Z_L; then the load current is

$$\dot{I}_L = \dot{E}_1/(Z_L + Z),$$

where Z is the equivalent impedance of the transformer itself, supposed to be known. The load voltage is

$$\dot{E}_L = \dot{E}_1 - Z\dot{I}_L = \dot{E}_1 Z_L/(Z_L + Z) = \dot{E}_1/[1 + (Z/Z_L)].$$

Having expressed all the known and unknown quantities in the complex form, in either Cartesian or polar coördinates, the magnitude and direction of $\dot{E}_L$ can be determined, by using the general method, *i.e.*, equating the real and the imaginary parts on both sides of the equation.

Prob. 6. The equation for $\dot{E}_L$ given in the preceding problem leads to involved numerical computations. Moreover, the difference $E_1 - E_L$ cannot be accurately determined in this way when E_L differs but little from E_1. Show how to simplify the numerical work, by taking advantage of the fact that Z is small compared with Z_L. Solution: When a quantity a is small compared to unity, we have by division $1/(1 + a) = 1 - a + a^2 -$ etc. We have accordingly

$$\dot{E}_L = \dot{E}_1[1 - (Z/Z_L)] \text{ approximately,}$$

or

$$\dot{E}_L = \dot{E}_1 - \dot{E}_1 Z/Z_L. \quad \ldots \ldots \ldots \quad (A)$$

Let $\dot{E}_L$ be the vector of reference; consequently $\dot{E}_1 = E_1(\cos\theta + j\sin\theta)$. Let also $Z_L = z_L(\cos\phi_L + j\sin\phi_L)$ and $Z = z(\cos\phi + j\sin\phi)$. Then according to eqs. (154) and (156),

$$\dot{E}_1 Z/Z_L = E_1 z/z_L[\cos(\theta + \phi - \phi_L) + j\sin(\theta + \phi - \phi_L)],$$

or, denoting $\dot{E}_1 Z/Z_L$ by $\Delta\dot{E}_1$ and $\phi - \phi_L$ by β, we have

$$\Delta\dot{E}_1 = \Delta E_1[\cos(\theta + \beta) + j\sin(\theta + \beta)].$$

Equation (A) may now be written in the form

$$E_1(\cos\theta + j\sin\theta) = E_L + \Delta E_1[\cos(\theta + \beta) + j\sin(\theta + \beta)],$$

where

$$\Delta E_1 = E_1 z/z_L$$

is a known quantity, as well as the angle $\beta = \phi - \phi_L$. Separating the real and the imaginary parts, we get

$$E_1\cos\theta = E_L + \Delta E_1\cos(\theta + \beta), \quad \ldots \ldots \ldots \quad (B)$$

$$E_1\sin\theta = \Delta E_1\sin(\theta + \beta) = \Delta E_1\sin\theta\cos\beta + \Delta E_1\cos\theta\sin\beta. \quad (C)$$

[1] The conditions in this problem differ from those in the text above in two respects: (1) The primary voltage is given instead of the secondary; (2) the load is given by its impedance instead of the current and power-factor.

From eq. (C), dividing both sides by $\cos\theta$, we find

$$\tan\theta = \Delta E_1 \sin\beta/(E_1 - \Delta E_1 \cos\beta), \quad . \quad . \quad . \quad . \quad \text{(D)}$$

from which θ can be calculated. Using in eq. (B) the transformation $\cos\theta = 1 - 2\sin^2 \frac{1}{2}\theta$, the same as in Art. 33, we get, after division by E_1,

$$(E_1 - E_L)/E_1 = (\Delta E_1/E_1)\cos(\theta + \beta) + 2\sin^2 \tfrac{1}{2}\theta. \quad . \quad . \quad \text{(E)}$$

While the derivation of formulæ (D) and (E) may seem somewhat tedious, the results are in the form most convenient for numerical work.

41. Analytical Determination of Voltage Regulation.—Exact Solution.[1] The approximation made in the preceding article consists in shifting the exciting admittance Y_0 so that it is connected across the primary terminal voltage E_1, instead of across the primary induced voltage E_{i1} (compare Figs. 39 and 41). Retaining the exciting admittance in its correct place, we obtain the equivalent diagram shown in Fig. 42. The secondary im-

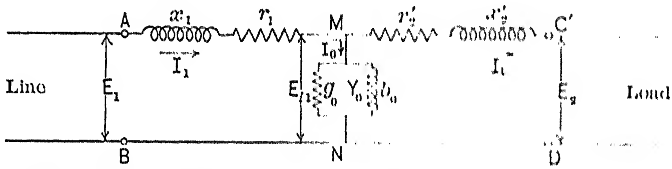

FIG. 42. The correct equivalent diagram of a transformer or an induction motor.

pedance is reduced to the primary circuit as before, by being multiplied by the square of the ratio of turns $(n_1/n_2)^2$. This procedure is strictly correct, the magnetic link being by assumption perfect.

Equations (186) and (187) hold here as before, but instead of using the approximate eq. (191) we shall use the correct relation (189). The magnetizing current is

$$I_0 = E_{i1}Y_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

so that the total primary current is

$$I_1 = I_L + E_{i1}Y_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

Expressing I_1 and I_2 in eqs. (186) and (187) through the load current I_L, and eliminating E_{i1} and E_{i2} as before, we obtain

$$(E_1 - Z_1 I_L)/(1 + Z_1 Y_0) = E_L + I_L (n_1/n_2)^2 Z_2. \quad (201)$$

This equation takes the place of the approximate eq. (192). The two equations become identical when $Y_0 = 0$.

[1] This article may be omitted if desired.

The complex quantity $1 + Z_1Y_0$ which enters into eq. (201) may be called a correction factor, and may be represented in the form

$$K = 1 + Z_1Y_0 = k(\cos\alpha + j\sin\alpha). \quad . \quad . \quad . \qquad (202)$$

Since the ratio of two complex quantities is also a complex quantity, eq. (201) may be expressed in the form

$$E_{c1} = E_L + I_LZ_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (203)$$

where the corrected primary voltage

$$E_{c1} = E_1/K = (E_1/k)\,[\cos(\theta - \alpha) + j\sin(\theta - \alpha)], \qquad (204)$$

and the corrected equivalent impedance

$$Z_c = Z_1/K + (n_1/n_2)^2Z_2 = z_c(\cos\psi + j\sin\psi). \quad . \qquad (205)$$

Equation (203) is of the same standard form as eqs. (195) and (159), and can be solved by the method given in Art. 33.[1]

This problem can be solved also by keeping the complex quantities in the orthogonal form. The student will profit by working out the details for himself.

Sometimes it is desired to know the voltage regulation of a transformer over a certain range of loads, the results being represented in the form of a curve. In such a case, it makes no difference for which particular loads the regulation is actually calculated, provided that these loads are selected within certain limits. If the primary voltage is given and is constant, it may be more convenient to perform the calculations (according to Fig. 42), not for an assumed current I_2, but for an assumed load impedance Z_L. Combining the impedances in series and the admittances in parallel, the whole circuit connected at the primary terminals is finally reduced to one impedance. Dividing the primary voltage by this impedance gives the primary current, and consequently the drop in the primary impedance Z_1. Thus, the voltage E_{i1} becomes known, and the current I_0 can be calculated. After this, the current I_L and the drop I_LZ_2 are calculated. This drop, being subtracted from E_{i1}, gives the desired secondary voltage E_2, reduced to the primary circuit.

[1] Namely, when $I_L = 0$, $E_L = E_{c1}$, so that per cent regulation is equal to $100\,\Delta E/(E_{c1} - \Delta E)$, where $\Delta E = E_{c1} - E_L$ (algebraically, not geometrically). Consequently, eq. (166) and the table on page 96 are directly applicable.

CHAPTER XII

PERFORMANCE CHARACTERISTICS OF THE INDUCTION MOTOR

42. The Equivalent Electrical Diagram of an Induction Motor. The student is supposed to be familiar with the general (qualitative) explanation of the performance of a polyphase induction motor, and with the general shape of the load characteristics.[1] It will be shown here how to predetermine the performance characteristics of a given induction motor by reducing it to an equivalent electric circuit, similar to that of a transformer.

The following experiment shows the possibility of such an equivalent diagram. A brake test is performed on the motor, and the primary current and the power-factor are plotted against the output as abscissæ. Then the rotor is blocked, and variable non-inductive resistances are inserted into its phase windings. If the rotor has a squirrel-cage secondary, resistances must be inserted in series with each bar, or into each section of the end-rings between consecutive bars. The motor is thus reduced to a polyphase transformer, the inserted secondary resistances representing the load. A load test is performed on this transformer, and the curves of primary current and power-factor are plotted against the total i^2r loss in the external resistances. These curves are found to coincide very closely with the curves obtained from the brake test, provided that the brake power and the i^2r power are plotted to the same scale, one representing the mechanical, the other the corresponding electrical output. Some difference in the curves is due to the fact that the stationary transformer has no friction loss; this is, however, partly or wholly compensated by a greatly increased secondary core loss.

The theoretical reasons for this equivalence of an induction motor to a polyphase transformer will become clear by consider-

[1] See, for instance, the author's *Experimental Electrical Engineering*, Vol. 1, chap. 17.

ing the electrical relations in the rotor under the following three headings:

(a) *The Relationship between the External Resistance and the Slip.* Let the input into the rotor be P watts per phase of the secondary winding, and let the motor be running at a slip s. For instance, $s = 0.05$ means that the speed of the rotor is 5 per cent lower than the synchronous speed, or the speed of the revolving field. Then, sP watts are converted into heat in each phase of the secondary winding, and $(1 - s)P$ watts are available on the shaft as the output (including friction and windage). This is because the tangential electromagnetic effort is the same on the surface of the stator as it is on the surface of the rotor. But while the gliding magnetic flux travels at synchronous speed, the rotor travels at $(1 - s)$ times the synchronous speed. The electromagnetic coupling between the stator and the rotor is similar to a friction coupling between two shafts, having a certain amount of slip. If the speed of the driven shaft is say 5 per cent below that of the driving shaft, on account of the slip in the coupling, 95 per cent of the power is transmitted and 5 per cent is lost in heat in the coupling.

With the rotor blocked, let R be the external resistance per phase of the secondary, and let I_2 be the secondary current per phase. For a slip s we must have the condition

$$s\,(R + r_2)\,I_2^2 = r_2 I_2^2,$$

or

$$s = r_2/(R + r_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (206)$$

If the slip is given, the required external resistance is

$$R = r_2(1 - s)/s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (207)$$

(b) *Equal Secondary Current and Phase Displacement with the Rotor Running or Blocked.* Let the reactance of the secondary winding per phase be x_2 ohms, at the primary or synchronous frequency. With the rotor running at a slip s, the frequency of the secondary currents is only equal to s times the primary frequency, so that the reactance per phase is sx_2. Therefore, the phase displacement ϕ_2 between the induced secondary voltage and the current is determined by the relation

$$\tan \phi_2 = sx_2/r_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (208)$$

With the rotor blocked and provided with external resistances satisfying condition (206), the total resistance of the secondary

circuit per phase is $R + r_2 = r_2/s$. The secondary frequency is equal to that in the primary circuit, so that

$$\tan \phi_2 = x_2/(r_2/s) = sx_2/r_2;$$

thus the phase displacement is the same as that given by eq. (208). With the same revolving magnetic flux in both cases, the currents are also equal. While with the stationary rotor the induced secondary e.m.f. is larger in the ratio of $1 : s$, because of a higher speed of cutting the secondary conductors, yet the total secondary resistance $R + r_2$ is also larger in the same ratio of $1 : s$, according to eq. (206). The secondary reactance is also larger in the same ratio on account of the higher frequency. Thus, with the rotor blocked, both the e.m.f. and the impedance of the secondary circuit are larger in the ratio of $1 : s$ than when it is running at a slip s. Hence, the current, which is equal to the ratio of the e.m.f. to the impedance, is the same in both cases.

(c) *The Reaction of the Secondary upon the Primary Circuit is the Same with the Rotor Running or Blocked.* The magnetomotive force of the revolving rotor is the same as that of the stationary rotor with the resistance R in series, provided that in both cases the magnetomotive force is considered *with respect to the stationary primary circuit.* In the latter case the frequency of the secondary currents is equal to that of the supply, so that the resultant magnetomotive force due to all the secondary phases travels in the air-gap at synchronous speed, the same as the resultant magnetomotive force of the primary currents. The two magnetomotive forces form one resultant magnetomotive force which produces the revolving flux. With the revolving rotor, the frequency of the secondary currents is s per cent of that of the supply, so that the secondary magnetomotive force glides relatively to the body of the rotor at a speed equal to s per cent of the synchronous speed. But the speed of the rotor itself is the $(1 - s)$ part of the synchronous speed. Hence, the velocity of the secondary magnetomotive force *with respect to the stator* is $s + (1 - s) = 1$, or is equal to the synchronous speed, and is the same as the velocity of the primary magnetomotive force. We have seen above that the secondary currents and their phase relation are the same in the two cases, so that the secondary magnetomotive force is also the same in phase and magnitude. Consequently, with the same flux, determined by the applied volt-

age, the primary magnetomotive force is also the same in both cases. This means that the primary current and power-factor are the same with the stationary rotor loaded electrically as with the revolving rotor loaded mechanically.

We have thus proved theoretically, as well as experimentally, that the performance of an induction motor may be reduced to that of a stationary transformer. But we know from the preceding chapter that a transformer can in turn be replaced by an equivalent electric circuit, either approximately (Fig. 41) or accurately (Fig. 42). Thus, the same equivalent diagrams can be used in the predetermination of the performance of an induction motor. All the quantities which enter into these diagrams are understood to be per phase of the primary circuit (usually per phase of Y in a three-phase motor). When the number of the secondary phases and the method of connections are different from those in the primary circuit, the secondary winding is replaced by an equivalent one of the same number of phases, and with the same kind of connections as in the primary circuit; see Art. 45 below.[1]

Prob. 1. Explain the principle of the speed control of an induction motor by means of adjustable external resistances in the secondary circuit.

Prob. 2. Explain the principle of the direct and differential cascade connection of two induction motors.

Prob. 3. Show how an induction generator can be reduced to an equivalent electric circuit.

43. The Analytical Determination of Performance. — Approximate Solution. The problem is to calculate the performance characteristics of a given induction motor — in other words, against the output as abscissæ, to plot the following curves; viz., primary amperes, kilowatts input, primary power-factor, slip, torque, and efficiency. The resistances and the leakage reactances of both windings, reduced to the primary circuit, are supposed to be known, so that each primary phase of the motor can be replaced by either the approximate or the exact diagram (Figs. 41 and 42). The approximate diagram only is considered here, because it is sufficiently accurate for most practical purposes. The exact solution is given in Art. 47 below. The iron loss, friction, and the

[1] The values of the leakage reactances of the windings are supposed here to be known; for their calculation from the dimensions of the motor see the author's *Magnetic Circuit*, Art. 66.

magnetizing current are also supposed to be known, so that the exciting admittance Y_0 is known. While in reality it varies somewhat with the load, in the approximate solution it is considered to be a constant quantity.

The problem is solved similarly to that of the voltage regulation of a transmission line or of a transformer, treated above; that is, a load current I_L is selected, and the circuit is solved in the complex notation. In order to make the treatment independent of the other chapters, a complete solution is given below, with some minor changes which simplify the numerical work. As in the transmission line and in the transformer, we have

$$\dot{E}_1 = E_L + Z\dot{I}_L, \quad \ldots \quad (209)$$

or, expanded,

$$E_1 (\cos\theta + j\sin\theta) = E_L + I_L (r + jx). \quad \ldots \quad (210)$$

Here the direction of the unknown load voltage E_L is again selected as the reference axis. The current I_L is in phase with E_L, because by assumption the external resistance R is non-inductive. Separating the real and the imaginary parts, we get

$$E_1 \cos\theta = E_L + I_L r; \quad \ldots \quad (211)$$

$$E_1 \sin\theta = I_L x. \quad \ldots \quad (212)$$

When plotting the curves, it is immaterial which values of the load are selected for computation. We assume, therefore, a series of reasonable values for I_L, and from eq. (212) calculate the corresponding values of $\sin\theta$. Then from eq. (211) we find the values of E_L, and finally determine the outputs per phase from the equation

$$P_L = I_L E_L. \quad \ldots \quad (213)$$

Knowing I_L, E_L, and the angle θ, the rest of the values for the performance curves are calculated as follows:

(a) *The Slip.* The external resistance, reduced to the primary circuit, is

$$R' = E_L / I_L; \quad \ldots \quad (214)$$

and the slip is found from the equation

$$s = r_2' / (R' + r_2'), \quad \ldots \quad (214a)$$

which is identical with eq. (206), except that r_2' and R' are secondary quantities reduced to the primary circuit.

(b) *The Primary Current and Power-factor.* The total primary current per phase is

$$\dot{I}_1 = \dot{I}_0 + \dot{I}_L, \quad \ldots \quad (215)$$

where the magnetizing current $\dot{I}_0$ is known, and can be represented with respect to the terminal voltage E_1 as

$$\dot{I}_0 = I_0 (\cos \phi_0 - j \sin \phi_0). \quad . \quad . \quad . \quad . \quad (216)$$

The load current, in its phase relation with respect to the terminal voltage, is

$$\dot{I}_L = I_L (\cos \theta - j \sin \theta), \quad . \quad . \quad . \quad . \quad . \quad (217)$$

so that

$$\dot{I}_1 = i_1 - ji_1' = (I_L \cos \theta + I_0 \cos \phi_0) - j (I_L \sin \theta + I_0 \sin \phi_0). \quad (218)$$

Knowing the projections i_1 and i_1' of the primary current with respect to the terminal voltage E_1, the primary phase angle ϕ_1 is found from the equation

$$\tan \phi_1 = i_1'/i_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (219)$$

and then the primary power-factor, $\cos \phi_1$, is taken from a trigonometric table. The current itself,

$$I_1 = i_1/\cos \phi_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

(c) *The input per phase* is

$$P_1 = E_1 I_1 \cos \phi_1 = E_1 i_1. \quad . \quad . \quad . \quad . \quad . \quad (221)$$

The efficiency is equal to the ratio of the output to the input.

(d) *The useful torque* in synchronous watts, or the input into the secondary, is equal to the output plus the secondary copper loss. The tangential effort per phase, in kilograms at a radius of one meter, or the torque per phase, in kg.-meters is

$$T = 973.8 \, (P_L + 0.001 \, I_L^2 r_2')/(\text{synchr. r.p.m.}), \quad . \quad (222)$$

P_L being expressed in kilowatts, so as to avoid large numbers in numerical applications.

Sometimes the performance data are desired for one particular load, P_L, only; for instance, at the rated output of the machine. The method outlined above may in this case be somewhat tedious, because one has to find by trials the proper values of I_L and E_L which give the desired output. It may lead more quickly to the desired end to solve eqs. (211), (212), and (213) as three simultaneous equations for the unknown quantities E_L, I_L, and θ. Squaring the first two equations and adding them together, the angle θ is eliminated, and we get

$$E_1^2 = E_L^2 + I_L^2 z^2 + 2 \, E_L I_L r.$$

Substituting for E_L its value from eq. (213), gives a quadratic equation for zI_L^2, namely

$$(zI_L^2)^2 - 2 \, (zI_L^2) \, (\tfrac{1}{2} \, E_1^2 - P_L r)/z + P_L^2 = 0. \quad . \quad . \quad (223)$$

The solution of this equation is

$$zI_L^2 = (\tfrac{1}{2}E_1^2 - P_Lr)/z - \sqrt{(\tfrac{1}{2}E_1^2 - P_Lr)^2/z^2 - P_L^2}.$$

The minus sign only is retained before the radical, because it gives a smaller current. It can be shown that the solution with the plus sign corresponds to the unstable region of operation of the motor.

For numerical computations the preceding equation is put in the form

$$zI_L^2 = 1000\,(Q - \sqrt{Q^2 - P_L^2}), \quad . \quad . \quad . \quad (224)$$

where, for the sake of brevity, we introduce the notation,

$$Q = (500\,E_1^2 - P_Lr)/z. \quad . \quad . \quad . \quad . \quad . \quad (225)$$

In the last two equations P_L and Q are in kilowatts, so as to avoid large numbers, and E_1 is in kilovolts. The student is reminded that E_1 is the phase or star voltage, and not the line voltage, and that P_L is the output per phase.

When P_L is small compared to Q, formula (224) represents the difference of two quantities of nearly equal value. The result is inaccurate, and it is better to expand the expression $[1 - (P_L/Q)^2]^{\frac{1}{2}}$, according to the binomial theorem. This gives

$$zI_L^2 = 1000\,Q\,[\tfrac{1}{2}(P_L/Q)^2 + \tfrac{1}{8}(P_L/Q)^4 + \tfrac{1}{16}(P_L/Q)^6 + \text{etc.}]. \quad (226)$$

The latter formula is much more convenient for numerical applications than eq. (224), because the second term in the brackets is small as compared to the first, and the third term can usually be neglected.

Knowing I_L, the rest of the values are determined as before.

Prob. 1. Plot complete performance curves of a three-phase, 25-cycle, 150-kw., 6-pole, 2200-volt, induction motor between no load and 25 per cent overload, from the following data: Total no-load input (for all three phases) is 10.5 kw.; the no-load current per phase of the line is 13 amp. With the armature blocked, the input is 230 kw., the current per phase being 227 amp.[1] The resistance of the primary winding per phase of Y is 0.60 ohm. Hint: Follow consistently the approximate diagram, Fig. 41; that is, do not correct the no-load reading for the primary i^2r loss, and assume the magnetizing current with the armature locked to be the same as at no load. See Art. 46, Prob. 2.

[1] The data with the armature locked refer to the rated voltage; they are obtained by extrapolating the curves taken at lower voltages. It would not be practicable to apply the full line voltage to a large motor with the armature blocked.

Ans. At the rated output the primary current is 49 amp.; the power-factor is 90.8 per cent; the slip, 3.5 per cent; the efficiency, 88.5 per cent; and the torque, 302 kg.-m.

Prob. 2. Check the answer to the preceding problem, using eq. (226).

Prob. 3. Extend the theory and the formulæ given above to the performance characteristics of an induction generator.

44. Starting Torque, Pull-out Torque, and Maximum Output. When judging the performance of a given induction motor, or designing a new motor, the following features are of importance:

(a) The starting torque, either in its absolute value, or in its ratio to the torque at the rated load. If the motor is to be started by means of resistances in the secondary circuit, one may be required to calculate the values of these resistances necessary for a prescribed starting torque, or for a maximum starting torque.

(b) The pull-out torque, or the torque at which the motor reaches the limit of stable operation, and comes to a stop. This torque is usually given through its ratio to the full-load torque.

(c) The maximum output of the motor, in kilowatts. This output takes place at a smaller slip than that at which the motor pulls out. The output is a maximum when the product of torque times speed is a maximum, but not when the torque itself is greatest.

The three quantities mentioned above can be determined by using the equations deduced in the foregoing article.

(a) *The Starting Torque.* In the general formula (222), $P_L = 0$ at start, because the motor supplies no mechanical output, the speed being equal to zero. In the equivalent electrical diagram (Fig. 41) this corresponds to a short circuit of the load, or $R = 0$. Hence, $I_L = E_1/z$; substituting this value into eq. (222), we find that the starting torque per phase, in kg.- m.

$$T_{st} = 0.9738\, E_1^2 r_2' / (z^2 \times \text{synchr. r.p.m.}). \qquad (227)$$

It will be seen from this expression that the starting torque is proportional to the square of the line voltage. This fact permits one to determine the initial torque when starting a motor on a lower voltage, by means of auto-transformers. The same equation shows that the starting torque increases with the secondary resistance. It is not quite proportional to it, because r_2' is also implicitly contained in z.

When the motor has a phase-wound secondary and is started by means of resistances in the secondary circuit, r_2' in formula (227) includes this secondary resistance. It is thus possible to calculate the external resistance required for a given starting torque. For example, when the starting torque is given, the ratio z^2/r_2' in eq. (227) is a known quantity, or

$$z^2/r_2' = [(r_1 + r_2')^2 + x^2]/r_2' = c, \quad . \quad . \quad . \quad (228)$$

where

$$c = 0.9738\, E_1^2/(\text{desired torque} \times \text{synchr. r.p.m.}). \quad (229)$$

Solving the quadratic (228) for r_2', we obtain

$$r_2' = (\tfrac{1}{2}\, c - r_1) \pm \sqrt{(\tfrac{1}{2}\, c - r_1)^2 - (x^2 + r_1^2)}. \quad . \quad (230)$$

In applications, the minus sign only is retained before the radical because one would naturally use the smaller of the two resistances which give the same torque.

The value of r_2' determined from this expression comprises both the resistance per phase of the rotor proper and the starting resistance per phase, if any is used, both reduced to equivalent primary values. To obtain their actual values, see Art. 45 below.

If the external resistance is to be so selected as to give a maximum starting torque, c in expression (228) must be a minimum. Equating to zero the derivative of c with respect to r_2', and solving for r_2', we get

$$r_2' = (x^2 + r_1^2)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (231)$$

that is, r_2' is very nearly equal to x. This value comprises the resistance of the rotor proper and the starting resistance, both per phase of the primary circuit. If a resistance is selected which is either less than or greater than that determined by eq. (231), the motor does not develop its full starting torque. This checks with eq. (230), which shows that the same torque can be obtained with two different values of starting resistance.

(b) *Pull-out Torque.* According to eq. (222), the torque is a maximum when

$$E_L I_L + I_L^2 r_2' = \text{max.} \quad . \quad . \quad . \quad . \quad (232)$$

Here E_L and I_L are functions of the independent variable θ. Expressing them through θ from eqs. (211) and (212), and omitting the constant factor E_1^2, we obtain

$$(1/x) \sin\theta\, [\cos\theta - (r/x) \sin\theta] + (r_2'/x^2) \sin^2\theta = \text{max.},$$

or, after simplification,

$$x \sin 2\theta - r_1 (1 - \cos 2\theta) = \text{max.} \quad . \quad . \quad . \quad (233)$$

Equating to zero the derivative of this expression with respect to θ, gives

$$x \cos 2\theta - r_1 \sin 2\theta = 0,$$

or

$$\tan 2\theta = x/r_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

Knowing θ, the values of E_L and I_L are calculated from eqs. (211) and (212), and then the torque is determined from eq. (222).

It is of interest to note that the angle θ, at which the motor pulls out of step, is independent of the secondary resistance r_2'. Neither does this resistance enter into eq. (233). Hence, *the maximum torque which a motor is capable of developing is independent of its rotor resistance. This resistance determines only the speed at which the maximum torque takes place.* The higher the secondary resistance, the lower the speed at which the motor pulls out of step. By using an external starting resistance, and a rotor winding of low resistance, two maxima of the torque are obtained, one at the start, with the external resistance in, and the other near synchronism, with it out.

(c) *Maximum Output.* The problem is to find the values of E_L and I_L for which the product $E_L I_L$ is a maximum. Again expressing E_L and I_L through the angle θ from eqs. (211) and (212), and omitting the constant factor E_1^2/x, we have

$$\sin\theta\,[\cos\theta - (r/x)\sin\theta] = \text{max.}$$

Equating to zero the derivative of this expression with respect to θ, gives

$$\theta = \tfrac{1}{2}\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (235)$$

where $\tan\phi = x/r$. Knowing the angle θ, the values of I_L and E_L are calculated from eqs. (211) and (212), and then their product $E_L I_L$ is determined.

Prob. 1. The motor specified in problem 1 of the preceding article is designed to be started at a reduced voltage. What per cent tap should be used on the auto-transformers in order to get a starting torque of about 30 per cent of the full-load torque?

Ans. 60 per cent of the line voltage.

Prob. 2. The same motor is provided with a phase-wound secondary and is to be started by using resistances in series with the rotor windings.

What external resistance is necessary in order to obtain a starting torque equal to 1.5 times the full-load torque?

Ans. 0.78 ohm per phase, in terms of the primary circuit.

Prob. 3. What starting resistance in the preceding problem would give the maximum starting torque? Ans. About 5.44 ohms.

Prob. 4. Show that the motor specified in problem 1 of the preceding article pulls out of step when the torque exceeds 2.45 times the rated full-load torque.

Prob. 5. Check the answer to problem 4 by using the answer to problem 3.

Prob. 6. Show that the maximum output of the same motor is equal to 2.15 times the rated output.

Prob. 7. Show that the input into an induction motor is a maximum when $\theta = 45°$. Hint: $I_L \cos \theta = \max.$

Prob. 8. Show how to calculate the per cent slip at which the motor pulls out of step, and also the speed at which the output is a maximum.

CHAPTER XIII

PERFORMANCE CHARACTERISTICS OF THE INDUCTION MOTOR—(*Continued*)

45. The Secondary Resistances and Reactances Reduced to the Primary Circuit. It is proved in Art. 40 that in a transformer the secondary resistance and reactance can be transferred into the primary circuit by being multiplied by $(n_1/n_2)^2$. The same rule holds true for the induction motor, provided that the number of phases is the same in the primary and in the secondary windings, and that the two windings are of the same type (the same number of slots per phase and the same winding pitch). This is hardly ever the case, and with a different number of phases and different types of winding in the primary and secondary, the following formulæ hold true:

$$r_2'/r_2 = (m_1/m_2)\,(k_{b1}n_1/k_{b2}n_2)^2 \quad . \quad . \quad . \quad . \quad (236)$$

$$x_2'/x_2 = (m_1/m_2)\,(k_{b1}n_1/k_{b2}n_2)^2 \quad . \quad . \quad . \quad . \quad (237)$$

In these expressions, m is the number of phases, n is the number of turns per pole per phase, and k_b is the so-called breadth factor which characterizes the winding. The subscripts 1 and 2 refer to the primary and secondary windings respectively. The quantities r_2 and x_2 are the actual resistance and reactance per pole per phase of the secondary circuit; r_2' and x_2' are the equivalent quantities per pole per phase of the primary circuit.[1] When $m_1 = m_2$ and $k_{b1} = k_{b2}$, the preceding formulæ become identical with eqs. (196) and (196a) for the transformer.

Formula (236) is a combination of the following two equations:

$$0.9k_{b2}m_2n_2i_2 = 0.9k_{b1}m_1n_1i_2' \quad . \quad . \quad . \quad . \quad (236a)$$

$$m_2i_2^2r_2 = m_1i_2'^2r_2' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (236b)$$

In these equations i_2 is the effective value of the current in any conductor of the actual secondary winding, and i'_2 is that in the

[1] For a proof of these formulæ, see the author's *Magnetic Circuit*, Art. 44; the values of k_b will be found in Arts. 27 to 29 of the same book.

equivalent rotor winding, transferred into the primary circuit. Eq. (236a) expresses the condition that the magnetomotive forces per pole of the actual rotor and equivalent rotor are equal to each other. Eq. (236b) states that the heat losses in the two windings are also equal. Eq. (236) is obtained by eliminating the ratio (i_2/i'_2) from the foregoing two expressions.

It is sometimes convenient to keep in mind this method of derivation of eq. (236) in order to apply this equation properly in a practical case. For example, one of the windings may be mesh connected, and the other star connected. One of the windings may have all the turns per phase in series, the other may consist of two or more branches in parallel, etc. Keeping in mind the physical conditions (236a) and (236b), the proper values of r_2 and r_2' can be determined in all such cases. A two-phase combination of currents is not a symmetrical system to which eq. (236a) can be applied directly. It is preferable to replace it by an equivalent quarter-phase system (Art. 35).

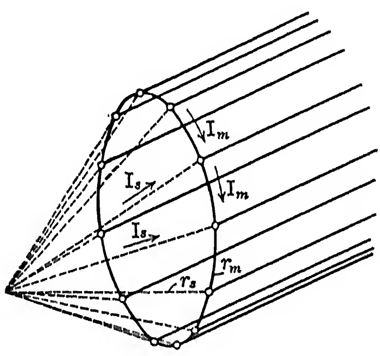

FIG. 43. A squirrel-cage rotor and the star resistances equivalent to the end ring.

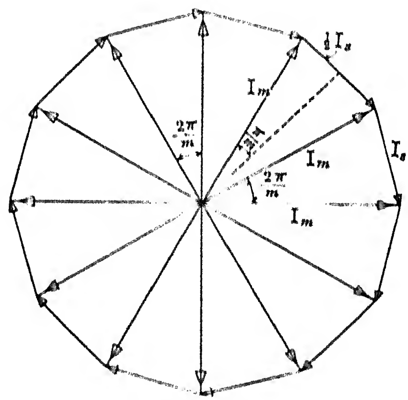

FIG. 44. The vectorial relation between the star and mesh currents in a symmetrical m-phase system.

In the most general case, let an m-phase symmetrical system be given, for instance a two-pole squirrel-cage rotor (Fig. 43), and let it be required to find the relation between the mesh resistances r_m and the star resistances r_s such that the i^2r loss per phase be the same in both. The relation between the vectors of the star currents and those of the mesh currents is shown in Fig. 44, the star currents I_s forming an m-sided polygon, and the mesh currents I_m being the radii of the polygon. This diagram satisfies the following conditions: (a) the star currents are displaced in

phase relatively to each other by equal angles of $2\pi/m$, over the whole range of 2π; (b) the same is true for the mesh currents; (c) each star current is the geometric difference of the two adjacent mesh currents; (d) the geometric sum of the star currents is equal to zero (Kirchhoff's first law). From the geometry of the figure we have

$$\tfrac{1}{2}I_s = I_m \sin(\pi/m) \quad \ldots \ldots \quad (238)$$

The condition $I_s^2 r_s = I_m^2 r_m$ leads to the ratio

$$r_m/r_s = (I_s/I_m)^2 = 4\sin^2(\pi/m) \quad \ldots \quad (239)$$

A similar relation holds for the reactances. When $m = 3$, we find that $r_\Delta/r_Y = 3$.

Equation (239) finds its practical application in the calculation of the equivalent resistance and reactance of a squirrel-cage rotor. The sections of the end-rings between the bars are mesh-connected, while the bars themselves may be considered as parts of a star-connected m_2-phase system, where m_2 is the number of bars per pair of poles. Let r_b be the resistance of each bar, including the two contact resistances between the end-rings and the bar; let r_r be the resistance of a section of an end-ring between two consecutive bars. The resistance of the rings can be replaced by added resistances in series with the bars, so as to change the connections to a pure star system (Fig. 43). According to eq. (239), the total equivalent resistance per bar is

$$r_{br} = r_b + \tfrac{1}{2}\, r_r/\sin^2(\pi/m_2) \quad \ldots \ldots \quad (240)$$

where r_{br} includes the resistance r_b of the bar itself, and of the adjacent sections, r_r, of both end rings. Analogously

$$x_{br} = x_b + \tfrac{1}{2}\, x_r/\sin^2(\pi/m_2). \quad \ldots \ldots \quad (241)$$

When applying eqs. (236) and (237) to a squirrel-cage rotor, the following secondary values should be used:

(1) The number of secondary phases is equal to the number of bars per pair of poles, that is, $m_2 = C_2/(\tfrac{1}{2}p)$, where C_2 is the total number of rotor bars. (2) There is but one bar per pair of poles per phase, and a bar is equivalent to one half of a turn; hence, $n_2 = 0.25$. (3) A squirrel-cage winding is a full-pitch winding of small breadth so that $k_{b2} = 1$. (4) In eq. (236) r_2 stands for the resistance per pole per phase, while in eq. (240) r_{br} is the resistance per bar. Since there is only one bar per pair of poles per phase, $r_2 = \tfrac{1}{2}\, r_{br}$.

Prob. 1. A two-phase induction motor has the primary winding arranged for two independent phases; the secondary is three-phase Y-con-

nected. When the rotor is stationary and its circuits are open, 440 volts impressed at the primary terminals produce 97 volts between the slip-rings. The calculated starting resistance, per phase of the primary circuit, is 14 ohms. What is the actual resistance to be used in series with the rotor windings? Hint: Consider the primary circuit as a four-phase star-connected system, so that $n_1k_{b1}/n_2k_{b2} = \frac{1}{2} \times 440 \,(97/\sqrt{3})$.

Ans. 0.34 ohm.

Prob. 2. A six-pole, three-phase, Y-connected induction motor has a squirrel-cage rotor of 80 cm. diameter with 73 bars; the resistance of each bar is 120 microhms (including the contact resistance). In order to have a certain required torque and slip, the equivalent rotor resistance per phase of the primary circuit must be equal to 1.07 ohms. What must be the actual resistance of each end-ring per centimeter of its length? There are 100 turns per phase of the primary winding, and $k_{b1} = 0.95$; for the squirrel-cage winding k_{b2} is always equal to unity.

Ans. 5.8 microhms.

46. The Circle Diagram. Let the values of primary current obtained from a brake test on an induction motor be plotted as vectors at proper phase angles with respect to the vector E_1 of the primary voltage (Fig. 45). The locus of the ends of the current vectors is found to be very nearly a semicircle. This

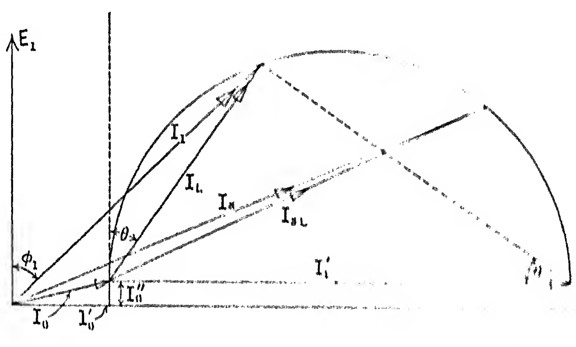

FIG. 45. The circle diagram of an induction motor.

locus, together with some auxiliary lines, is called the circle diagram or the Heyland diagram of the induction motor. A similar diagram holds true for the transformer, although it is hardly ever used in practice.

The importance and the convenience of the circle diagram lie in the fact that the complete performance of an induction motor can be predicted if the diameter of the semicircle and its posi-

tion with respect to the voltage vector E_1 are known. The semicircle is usually determined by the vector I_0 of the no-load current, and the vector of the current I_s obtained when the rotor is locked (the subscript s stands for starting or short circuit). At any other load, the extremity of the current vector I_1 lies between those of I_0 and I_s.

The Heyland diagram is simply a graphic representation of the current and voltage relations in the approximately equivalent circuit diagram shown in Fig. 41. The exciting current and the no-load losses are assumed to be constant at all loads from no-load to standstill. The no-load current is resolved into a loss component I_0'', in phase with the voltage E_1, which component represents the iron loss, friction, and windage; and a reactive component I_0' which excites the main flux in the motor. At any load, the primary current I_1 is the geometric sum of the load current I_L and the no-load current I_0, the vectors of these three currents forming a triangle.

That the locus of the current I_1 or I_L is a circle follows directly from eq. (212), because from it we have

$$I_L/\sin\theta = E_1/x = \text{const.}, \quad . \quad . \quad . \quad . \quad (242)$$

which is easily seen to be the equation of a circle in polar coördinates. The value of the constant

$$E_1/x = I_L' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (243)$$

is equal to the diameter of the circle, which diameter is thus determined solely by the leakage reactance x of the motor. The smaller x is, the larger is the circle and the better the motor, because its power-factor is higher and its overload capacity larger.

The load current I_{sL} with the armature blocked has an energy component in phase with the line voltage E_1, because of the i^2r loss in the resistances of the stator and rotor. If these resistances could be eliminated or put outside the motor, the load current on short circuit would be purely reactive and equal to $I_L' = E_1/x$. Thus, the diameter of the circle is equal in position and magnitude to the load current (secondary current) of the machine with the armature blocked, provided that the internal resistances are eliminated and only leakage reactances are left. This condition is called an ideal short circuit.

Leaving the loss component I_0'' of the no-load current out of

consideration, the performance of the motor is determined by the pure magnetizing current I_0' and its ratio to the diameter I_L' of the semicircle. This ratio is called the *circle coefficient*. Let the exciting reactance, or the reciprocal of b_0, be denoted by x_0. Then $I_0' = E_1/x_0$, and, by definition, the circle coefficient

$$\sigma = I_0'/I_L' = x/x_0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (244)$$

in other words, the circle coefficient is equal to the ratio of the leakage reactance to the exciting reactance.[1]

Thus, knowing the magnetizing current and the short-circuit current (or the magnetizing current and the circle coefficient), the Heyland circle can be drawn, and the relation between the primary current, the load current, the power-factor, and the angle θ graphically determined. By drawing certain auxiliary lines, the input, output, slip, torque, and efficiency can also be read off directly from the diagram, for any assumed primary current.[2] In other words, the circle diagram permits one to determine graphically the performance characteristics, and offers an alternative method to the analytical procedure explained in Art. 43 above. The relative advantages of the analytical and graphical methods depend upon the problem in hand and the skill and temperament of the user; the student should thoroughly familiarize himself with both methods before deciding upon the use of one or the other.

The circle coefficient is very convenient for preliminary designs and performance estimates. Mr. H. M. Hobart has made quite a study of the numerical values of this coefficient for a large number of actually built motors, and has compiled his results in the

[1] The circle coefficient is also called the dispersion factor (*Streuungskoeffizient*), and is usually denoted by σ. Those familiar with magnetic phenomena will notice that the circle coefficient is equal to the ratio of the permeance of the main magnetic path in the motor to that of the leakage paths. This is because the reactances are proportional to the corresponding inductances, and an inductance is equal to the permeance of the path times the square of the number of turns linked with it. A motor is evidently improved by reducing the permeance of its leakage paths and increasing that of the useful path. This means that the motor is better the lower its circle coefficient σ.

[2] For complete and explicit instructions in regard to the construction and use of the circle diagram, see the author's *Experimental Electrical Engineering*, Vol. 2, Chap. 29.

form of charts, from which the value of the coefficient may be taken for a motor of given or assumed dimensions.[1]

Prob. 1. Show that the circle coefficient of the motor specified in problem 1, Art. 43, is equal to 0.0572.

Prob. 2. Check a few points on the curves obtained in problem 1, Art. 43, by constructing the circle diagram of the motor.

47. The Analytical Determination of Performance.—Exact Solution.[2] The predetermination of the performance characteristics of an induction motor, explained in Arts. 43 and 46, is based upon the approximately equivalent diagram shown in Fig. 41. The exact performance characteristics are obtained by expressing analytically the electrical relations according to the correct equivalent diagram shown in Fig. 42. To be absolutely correct, both g_0 and b_0 must be varied somewhat with the load, because (a) the friction and windage depend upon the speed, (b) the iron loss is not exactly proportional to the square of the flux, and (c) the magnetizing current is not proportional to the voltage. Moreover, the friction loss ought to be separated from the iron loss, and subtracted from the output, instead of being added to the input. All these corrections make the calculations much more involved, and, while it is well to know about them, they are hardly ever justified in practice.

In large and medium-sized motors the losses and the internal voltage drop are comparatively small, so that the performance calculated according to the exact diagram differs but little from that obtained with much less time and effort, by using the approximate diagram. It is only in small motors, or where extreme accuracy is required for some special reason, that the procedure given below is justified. In very small motors, say below one kilowatt, the difference between the approximate and the correct performance is quite appreciable, because of high losses and a large voltage drop.

[1] H. M. Hobart, *Electric Motors* (1910), Chapter 21. It may be of interest to note that the correct equivalent diagram (Fig. 42) also leads to a circle diagram, known as the Ossanna circle. Numerous articles on this exact diagram will be found in the various volumes of the *Elektrotechnische Zeitschrift* and *Elektrotechnik und Maschinenbau*.

[2] This article may be omitted if desired, because the approximate solution given in Arts. 43 and 46 is sufficient in a great majority of practical cases. However, the method used in this article is of interest to the student as another and somewhat different application of complex quantities.

It is much more convenient to plot complete performance curves than to calculate the performance data for a *specified* output. A certain external resistance R' is assumed, such as would give a reasonable value of slip according to eq. (214a), and the performance characteristics are calculated for this value of R'. Then another value of R' is assumed, and the calculations are repeated, and so on. For an assumed value of R' the total admittance between the primary terminals is calculated, using the general method given in Art. 28, that is, adding impedances in series and admittances in parallel. Knowing the total admittance, the primary current becomes known; then I_L and E_L are calculated, and finally the rest of the data are obtained as in Art. 43. The details of the calculations are as follows:

(1) The impedance of the load plus that of the secondary winding $= (R' + r_2') + jx_2'$.

(2) Using eqs. (121) and (122), Art. 27, find the corresponding admittance $g_2 - jb_2$.

(3) The total admittance between the points M and N is $(g_0 + g_2) - j\,(b_0 + b_2)$.

(4) Using eqs. (123) and (124), Art. 27, find the corresponding impedance $r_{MN} + jx_{MN}$.

(5) The total impedance between the primary terminals is $Z_{eq} = (r_{MN} + r_1) + j\,(x_{MN} + x_1)$.

(6) The corresponding admittance $Y_{eq} = g_{eq} - jb_{eq}$ is calculated from eqs. (121) and (122).

(7) The primary current is

$$\dot{I}_1 = \dot{E}_1 Y_{eq}. \qquad (245)$$

(8) The voltage across MN

$$\dot{E}_{i1} = \dot{E}_1 - \dot{I}_1 Z_1 = \dot{E}_1(1 - Z_1 Y_{eq}). \qquad (246)$$

(9) The load current is

$$\dot{I}_L = \dot{I}_1 - \dot{I}_0 = \dot{I}_1 - \dot{E}_{i1} Y_0,$$

or, substituting the values of $\dot{I}_1$ and $\dot{E}_{i1}$ from eqs. (245) and (246),

$$\dot{I}_L = \dot{E}_1 [Y_{eq}(1 + Y_0 Z_1) - Y_0]. \qquad (247)$$

(10) The load voltage

$$\dot{E}_L = \dot{I}_L R'. \qquad (248)$$

The rest of the quantities are calculated in the same manner as in Arts. 43 and 44.

In numerical work, it is convenient to take E_1 along the reference axis. Having determined the value of Y_{eq}, computations are begun with the composite admittance in the brackets in eq. (247). Either the orthogonal expressions of the form $r + jx$ or the polar expressions of the form $z(\cos\phi + j\sin\phi)$ may be used, according to one's preference or familiarity with one or the other form. The student ought to be familiar with both forms. The trigonometric form is convenient for multiplication and division, while the Cartesian form is preferable in addition and subtraction. It may be advisable to use both forms in the same problem.

The calculator should avoid long algebraic expressions, performing numerical operations step by step. Much time is saved by arranging the consecutive steps in a table, so as to repeat the same operations mechanically for different values of R'. An irregularity of the values in a column is a sure indication of a numerical error.

Much time is also saved by intelligently discriminating between the principal terms and small correction factors in an expression. For instance, in eq. (247) Y_{eq} is large as compared to Y_0 and to $Y_{eq}Y_0Z_1$. It would be a waste of time to figure out the latter expression accurately, when, in all probability, the principal term will be affected only by its first significant figure. On the other hand, the principal term, Y_{eq}, must be calculated to a degree of accuracy at least equal to that desired in the result, if not to a higher degree. Considerable skill, experience, and judgment are necessary to determine the proper accuracy of computations in engineering problems. This is an art which grows by intelligent exercise, and it is never too early to begin practicing it. The rewards are time and mental energy saved for better things, while obtaining an accuracy which is commensurate with the desired result.[1]

[1] For a complete set of final formulæ for induction motor characteristics, see Arnold's *Wechselstromtechnik*, Vol. 5, part 1 (1909), pp. 65–78. A very slight inaccuracy is introduced there in the beginning, by neglecting the imaginary part in a complex quantity. See also Dr. Steinmetz's *Alternating-current Phenomena*, under "Induction Motor."

Prob. 1. Make out a table showing in detail the order of computations for a complete set of performance characteristics of an induction motor, according to the method developed above.

Prob. 2. Mark on the curve sheet obtained in problem 1, Art. 43, a few points determined according to the exact equivalent diagram, in order to see the inaccuracy resulting from the use of the approximate method.

CHAPTER XIV

THE DIELECTRIC CIRCUIT

48. The Electrostatic Field.[1] In the following discussion, it is assumed that the student knows the fundamental phenomena of electrostatics from his study of physics. The purpose of the treatment given here is to deduce the principal numerical relations which are of importance in electrical engineering. The electrostatic field is considered in this book from Faraday's point of view, *viz.*, as consisting of displacements of electricity, and stresses

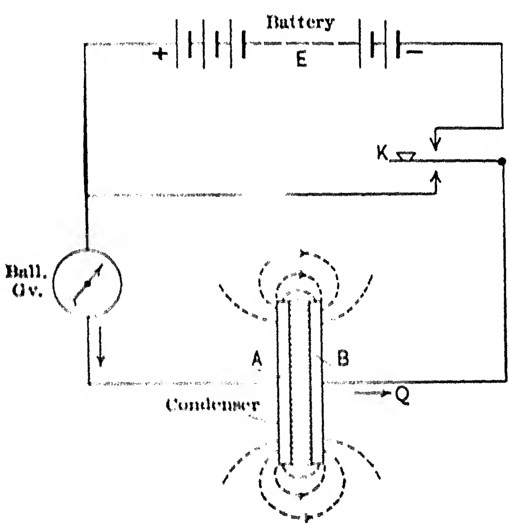

FIG. 46. A plate condenser completing a direct-current circuit.

in the dielectric. This is different from the older theory of the action of electric charges at a distance.

Let a source E of *continuous* electromotive force (Fig. 46) be connected to two parallel metallic plates A and B, the combination of which is commonly known as a *condenser*. Let the plates

[1] See the footnote at the beginning of Chapter 3.

be separated from each other by air, or by some other non-conducting material. When the key K is pressed upwards, a certain quantity of electricity, Q, flows from the battery to the plate A, and the same quantity flows from the plate B back to the battery. This quantity can be measured by the ballistic galvanometer shown in the circuit. Within a very short time the difference of potential between the plates becomes equal and opposite to that of the battery and the flow of current stops.

Since electricity behaves like an incompressible fluid, the same quantity, Q, is displaced through the whole circuit, including the layer of insulation or *dielectric* between the condenser plates. This displacement is accompanied by a stress in the dielectric, similar in some respects to a mechanical stress in an elastic body. The directions of the electric stress and of the lines of displacement of electricity through the air are shown in the figure by dotted lines. These stresses produce a counter-electromotive force, which finally balances that of the battery. When the key is opened, the condenser remains charged, since the stress and the displacement can be relieved only in a closed circuit. To discharge the condenser, its plates must be connected by a conductor; this is done by pressing the key down. The deflection of the ballistic galvanometer during the discharge is equal and opposite to that during the charge, and the electric energy stored in the condenser is dissipated by the current in the form of heat.

The difference between a dielectric and a conductor is that the resistance of the former to the passage of electricity is of an elastic nature; that is, the stress can be relieved and the stored energy returned to the circuit. On the contrary, the resistance to the flow of electricity in a conductor is of the nature of friction. The energy is converted into Joulean heat and cannot be restored.

The modern electronic theory of electricity is not sufficiently advanced at this writing to give a clear account of the true nature of these displacements and stresses in a dielectric. It is therefore preferable for our purposes not to specify the mechanism by which these stresses and displacements are produced. We shall simply assume, as a matter of fact, the structure of dielectrics to be such that an e.m.f. across a layer of such material produces a displacement of a certain quantity of electricity, which is proportional to the e.m.f. When the e.m.f. is removed and a closed circuit is provided, the stresses within the dielectric are relieved, and the

displacement disappears. The analogy to an elastic body subjected to external mechanical forces naturally suggests itself.

Experiment shows that, with given metallic plates (Fig. 46) and the same applied e.m.f., the value of the electric displacement depends upon the nature of the dielectric. With solid and liquid insulating materials, such as glass, oil, mica, etc., the same e.m.f. produces larger displacements of electricity than with air as the dielectric. These materials are therefore said to possess higher *permittivity* than the air (some writers use the word inductivity).

When an *alternating* voltage is applied at the terminals of a condenser, the displacement of electricity in the dielectric varies continually in its magnitude and periodically reverses its direction; consequently, it gives rise to an alternating current in the conducting part of the circuit. This is called the charging or capacity current. This current leads the alternating voltage in phase by 90 degrees, as may be seen from the following considerations: When the voltage has reached its instantaneous maximum the charging current is zero, because at the crest of the wave the voltage and the displacement remain practically constant for a short period of time. As soon as the voltage begins to decrease, the current begins to flow in the direction opposite to that of the applied voltage, because the elastic reaction of the dielectric is now larger than the applied electromotive force. At any instant, the current, or the rate of flow of electricity, is proportional to the rate of change of the applied voltage. But if the applied voltage varies according to the sine law, the rate of variation is also represented by a sine function differing in phase by 90 degrees from the original function, because $d(\sin x)/dx = \cos x = \sin(90^\circ + x)$; see also Art. 66 below. That there must be a displacement of 90 degrees between the voltage and the current follows also directly from the assumed elastic structure of the dielectric. The energy is supposed to be periodically stored in the dielectric and given up again without any loss; hence, the average power must be zero, and the current must be reactive.

49. A Hydraulic Analogue to the Dielectric Circuit. The hydraulic analogue shown in Fig. 47 may assist the student in the understanding of the electrostatic circuit. *A* is a pump which corresponds to the source of electromotive force in Fig. 46. The pipes *B* and *C* represent the leads to the condenser, or the metallic parts of the circuit. The cylinder *D* corresponds to the condenser,

and the elastic partition K is analogous to the dielectric. Let the pipes and the cylinders be filled with water, and let the piston in A be in its middle position, the partition K not being stressed. Let the stopcock M be open, and the stopcock N closed. When a

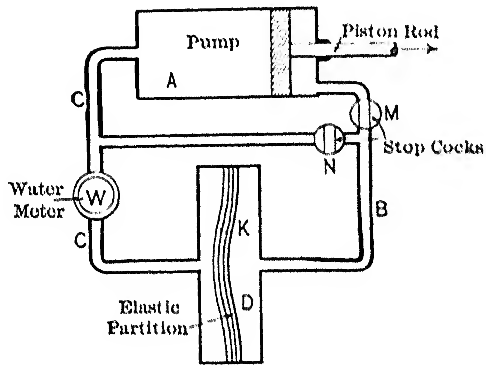

FIG. 47. A hydraulic analogue of a dielectric circuit.

pull to the right is exerted upon the piston rod and it is forced to move, the water in the system is displaced, and the elastic partition K is strained, as shown in the figure. With a given pull, or a given electromotive force, the movement stops when the pull is balanced by the elastic reaction of the partition. The charge, or the total displacement, is represented by the amount of water shifted; it can be measured by the water-meter W, which thus takes the place of the ballistic galvanometer.

If the pipes are frictionless, and the inertia of the piston and water is assumed negligible, the analogy can be followed still further; namely, the phase difference in time between the pull and the velocity of the water is equal to 90 degrees, the velocity leading the pull. Assuming the motion of the piston to be harmonic, the velocity of the flow of water is at its maximum when the piston is at the center of its stroke. The required pull is equal to zero at this moment, because the elastic partition is in its middle, or unstrained position. At the end of the stroke the velocity is zero, but the pull is at its maximum, because the partition is strained to its extreme position, and exerts its maximum elastic reaction. Thus the pull lags behind the velocity.

Substituting another partition, made of a more yielding

material (material possessing higher permittivity), a larger displacement is produced with the same pull; this corresponds to the case in which some solid or liquid dielectric is substituted for the air.

Closing the stopcock M corresponds to breaking the electric circuit of the condenser. It will be seen from analogy that the condenser remains charged. To discharge the condenser, the stopcock N must be opened; this equalizes the pressure on both sides of the elastic partition. Since, in reality, water possesses some inertia, the partition does not stop in its middle position during the discharge, but the momentum of the water carries it beyond the center. The electromagnetic inertia of the electric current produces a similar effect, and we thus have a simple explanation of the oscillatory character of the electric discharge. During this discharge, the energy is alternately transformed into the potential energy of dielectric stress, and into kinetic energy of the magnetic field. The oscillations of the partition are gradually damped out by the frictional resistance of the pipes. In the electric circuit, oscillations are damped by the ohmic resistance of the conducting parts of the circuit.

The student can follow this analogy still further, and explain free electrical vibrations, current and voltage resonance, also the effect of a resistance in series and in parallel with a condenser, etc.

50. The Permittance and Elastance of Dielectric Paths. Let Q (Fig. 46) be the total displacement of electricity in the dielectric, measured in ampere-seconds or coulombs, and let E be the voltage impressed across the condenser or "permittor." Experiment shows that up to a certain limit Q is proportional to E; this is similar to the behavior of an elastic body, in which the strains are proportional to the applied forces until the limit of elasticity has been reached. Thus, we may write

$$Q = CE, \qquad (249)$$

where the coefficient of proportionality, C, is called the *permittance* of the condenser. The older name for C is electrostatic capacity. When E is in volts and Q in coulombs, permittance is measured in units called *farads*. A condenser has a permittance of one farad when a displacement of one coulomb is produced for each volt applied at its terminals. The farad being too large a unit for practical use, permittances are usually measured in micro-

farads, one microfarad being equal to one millionth part of a farad.

The larger the permittance of a condenser, the larger is the displacement of electricity with the same voltage; hence C is a measure of the *ease* with which an electric displacement can be produced in a given condenser. In this respect the concept of permittance is analogous to those of electric conductance and magnetic permeance.

In some cases it is convenient to speak, not of the degree of ease, but of the *difficulty* with which an electric displacement can be produced in a given condenser. For this purpose, a coefficient of proportionality, the reciprocal of C, has to be used; and eq. (249) becomes

$$E = SQ, \qquad (250)$$

where

$$S = C^{-1} \qquad (251)$$

is called the *elastance* of the condenser. Elastance is thus analogous to electric resistance and to magnetic reluctance. When permittance is measured in farads, the unit of elastance is the reciprocal of the farad, and may therefore be properly called the *daraf*. This is a name derived by spelling the word farad backwards, that is, in the same way in which mho is derived from ohm.[1] A condenser has an elastance of one daraf when one volt of pressure is required for each coulomb of displacement within it. The farad being too large a unit for practical use, the daraf is consequently too small a unit. Therefore, in practice, elastances should be measured in megadarafs, one megadaraf ($= 10^6$ darafs) being the reciprocal of one microfarad.[2]

When two or more permittances are connected electrically in parallel, the resultant permittance is larger than that of any of the component condensers, because a larger path is offered to

[1] It may be of interest to mention in this connection a similar derivation of the name for a unit of magnetic reluctance. The henry being the natural unit of magnetic permeance (or inductance) in the ampere-ohm system, the author has proposed calling the corresponding unit of reluctance the *yrneh*, a word derived by spelling the word henry backwards. See his *Magnetic Circuit*, Art. 5.

[2] For a complete rational nomenclature of electric and magnetic quantities, see the table on page xii at the beginning of the book, and also the one in the Appendix.

the displacement. The relation is similar to that of conductances or permeances in parallel. Let C_1, C_2, etc., represent permittances connected in parallel across a source of constant voltage E, and let Q_1, Q_2, etc., be the corresponding electric displacements through these condensers (or permittors). Then, according to the definition of permittance, we have

$$\left.\begin{array}{l} Q_1 = C_1E \\ Q_2 = C_2E \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad \ldots \ldots \ldots \quad (252)$$

The equivalent permittance, C_{eq}, must be such as to allow of a displacement equal to the sum of the partial displacements, with the same voltage; hence,

$$\Sigma Q = C_{eq}E. \quad \ldots \ldots \ldots \quad (253)$$

Adding eqs. (252) together gives

$$Q_1 + Q_2 + \text{etc.} = E\,(C_1 + C_2 + \text{etc.}),$$

or, by comparison with eq. (253),

$$C_{eq} = \Sigma C. \quad \ldots \ldots \ldots \quad (254)$$

In other words, when permittances are connected in parallel, the equivalent permittance is equal to their sum.

When condensers (or elastors) are connected in series, it is more convenient to use their elastances. Since electricity behaves like an incompressible fluid, the displacement through several elastances in series is the same in all of them. Let this displacement be denoted by Q, and let the voltages across the terminals of the individual elastors be E_1, E_2, etc. Then,

$$\left.\begin{array}{l} E_1 = S_1Q \\ E_2 = S_2Q \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}, \quad \ldots \ldots \ldots \quad (255)$$

where S_1, S_2, etc., are the elastances of the separate condensers. The equivalent elastance must allow of the same displacement Q with the same total voltage, or

$$\Sigma E = S_{eq}Q. \quad \ldots \ldots \ldots \quad (256)$$

Adding eqs. (255) together gives

$$E_1 + E_2 + \text{etc.} = Q\,(S_1 + S_2 + \text{etc.}),$$

or, by comparison with eq. (256),

$$S_{eq} = \Sigma S. \qquad (257)$$

In other words, when elastances are connected in series, the equivalent elastance is equal to their sum. The analogy to the addition of conductances in parallel and resistances in series is self-evident (see Art. 3).

Prob. 1. A condenser, which has a permittance of 10 microfarads, is connected to a direct-current magneto, the speed of which is increased at a uniform rate, so that the voltage rises at a rate of 1.7 volts per second. Calculate the charging current.

Ans. 17 microamperes. Note: This is the principle of an apparatus used for measuring the acceleration of railway trains.

Prob. 2. An elastance of 10 kilodarafs is connected across a 220-volt, 50-cycle line. Show that the effective value of the charging current is 6.91 amp. Solution: The maximum displacement in the dielectric is $220\sqrt{2}/(10 \times 10^3) = 22\sqrt{2} \times 10^{-3}$ coulombs. This displacement is reduced to zero within $\frac{1}{200}$ of a second; hence, the average charging current is $4.4\sqrt{2}$ amp. The effective value, assuming a sine-wave of current, is $4.4\sqrt{2} \times (\frac{1}{2}\pi/\sqrt{2}) = 6.91$ amp.

Prob. 3. Show that with two condensers in parallel the ratio of the displacements equals that of the permittances or is inversely as the ratio of the elastances. What is the analogous relation for conductances and resistances?

Prob. 4. When two condensers are in series, show that the ratio of the voltage drops across them equals that of the elastances, or is inversely as the ratio of the permittances. What is the analogous relation for resistances and conductances?

Prob. 5. A sectionalized condenser, such as is used for calibration and exact measurements, is built up of the following permittances: 0.5, 0.2, 0.2, 0.05, and 0.05 microfarads. What is the extreme range of permittances and elastances possible by combining these sections in series and in parallel?

Ans. From 1 to 0.0192 mf., or from 1 to 52 mgd.

Prob. 6. Referring to the preceding problem, the sections of the condenser are connected as follows: 0.2, 0.05, and 0.05 mf. are in series, and the combination is shunted by 0.2 mf. Then the whole is put in series with 0.5 mf. Show that the resultant permittance is equal to 0.154 microfarads.

51. Permittivity and Elastivity of Dielectrics. Experiment shows that the permittance of a sample of any dielectric varies with its dimensions in the same way that the conductance of a metal or the permeance of a magnetic path in a non-ferrous medium does; namely, the permittance is proportional to the cross-section of the layer and inversely proportional to its length in the

direction of the lines of force. By increasing the cross-section of the path perpendicular to the lines of force (Fig. 46), the displacement is increased in the same proportion. On the other hand, the displacement is found to be inversely proportional to the thickness of the dielectric, since the distance through which the voltage must act is greater if the thickness is increased. These relations follow directly from the laws deduced in the preceding article for the addition of permittances in parallel and elastances in series. Thus, by analogy with eq. (21), Art. 5, we put

$$C = \kappa A/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (258)$$

where κ is called the *permittivity* of the dielectric. It is analogous to the conductivity of a conducting material, or to the permeability of a magnetic medium. Permittivity may be defined as the permittance of a cubic unit of dielectric, when the lines of displacement are straight lines perpendicular to one of its faces. For air the permittivity is

$$\kappa_a = 0.08842 \times 10^{-6} \text{ microfarads per cm. cube.} \quad . \quad (259)$$

For other dielectrics, liquid and solid, the permittivity is higher than that of air; that is to say, they are more yielding to an electromotive force. It is convenient to express their permittivities in terms of that of the air; for instance, we may say that the permittivity of a certain transformer oil is 2.1 times that of the air.

The *relative* permittivities of some important insulating materials are tabulated in Art. 56 below, merely to indicate their order of magnitude. For accurate values, the reader is referred to various published physical tables and engineering handbooks. The older name for relative permittivity is specific inductive capacity (or dielectric constant). It is more convenient in practice to use relative than absolute permittivities, because the necessity of tabulating small quantities like κ_a in eq. (259) is avoided. Besides, the data are more readily comparable with one another, and with the permittivity of air, which is a standard dielectric. This procedure is analogous to tabulating the conductivities of various metals in terms of that of pure copper, taken as 100 per cent. The absolute permittivity of a material is obtained by multiplying the absolute permittivity of air by the relative permittivity of the dielectric in question. Equation (258) thus becomes

$$C = K\kappa_a A/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (260)$$

where K stands for the relative permittivity.

The elastance of a prismatic piece of dielectric, with the lines of displacement parallel to one set of its edges, is expressed by analogy with eq. (20), Art. 5, as

$$S = \sigma l/A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (261)$$

where

$$\sigma = \kappa^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (262)$$

is called the *elastivity* of the dielectric. Elastivity is analogous to the resistivity of a conducting material or to the reluctivity of a magnetic medium, and may be expressed for practical purposes in megadarafs per centimeter cube. For air, the absolute elastivity is, according to eq. (259),

$$\sigma_a = \kappa_a^{-1} = 11.3 \times 10^6 \text{ megadarafs per cm. cube.} \quad . \quad . \quad (263)$$

The concept of relative elastivity could be introduced if necessary, in which case its values would be equal to the reciprocals of the relative permittivities tabulated in Art. 56. However, it is sufficient to use the relative permittivity, even when dealing with elastances, so that eq. (261) becomes

$$S = (\sigma_a/K)l/A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (264)$$

The nomenclature used above is due to Mr. Heaviside;[1] it is consistent and uniform with the nomenclature used in the electro-conducting and magnetic circuits, and is suggestive as to the nature of the phenomena. The electrostatic nomenclature now in general use comprises but three terms; namely, condenser, capacity, and specific inductive capacity. It is hoped that the more rational and complete nomenclature used here will help to a clearer understanding of the dielectric circuit, and will simplify engineering calculations relating thereto.[2]

Note: The author considers the above-given value of κ_a, eq. (259), to be an experimental coefficient, in the same sense in which other properties of materials are characterized by experimental coefficients. For an engineer, the volt and the ampere are arbitrary units established by an international agreement, no matter what their relation to the so-called absolute units. The value of κ_a can be calculated theoretically, assuming the ratio between the electrostatic and the electromagnetic units to be known. In the absolute electrostatic system of units, with air as the dielectric, a plate condenser having an area of A sq. cm. and a distance between the plates equal to l cm., has a capacity equal to $A/(4\pi l)$. The

[1] O. Heaviside, *Electromagnetic Theory* (1894), Vol. 1, p. 28.

[2] See the author's paper "Sur Quelques Calculs Pratiques des Champs Electrostatiques," in the Transactions of the *Congresso Internazionale delle Applicazioni Elettriche*, Turin, 1911.

factor 4π enters on account of an unfortunate selection of the expression for Coulomb's law, which should have been $q_1q_2/4\pi r^2$, instead of q_1q_2/r^2. In the absolute electromagnetic units the same capacity is equal to $(A/4\pi l)(3 \times 10^{10})^{-2}$, where 3×10^{10} is the velocity of light in centimeters per second. To obtain the result in microfarads, the foregoing expression must be multiplied by 10^{15}. On the other hand, the same capacity expressed in the rational units defined above is $\kappa_a A/l$. Equating the two expressions gives $\kappa_a = 10^{-5}/(9 \times 4\pi) = 0.08842 \times 10^{-6}$ microfarads per centimeter cube.

The fact that κ_a can be expressed through the velocity of light does not make κ_a the less an empirical coefficient, because the velocity of light itself is determined experimentally. As a matter of fact, one of the ways in which the velocity of light is determined consists in calculating it indirectly from the value of κ_a obtained from measurements.

Prob. 1. Show that in the English system $\kappa_a = 0.2244 \times 10^{-6}$ microfarads per inch cube.

Prob. 2. A condenser (Fig. 46) consists of two metal plates, 50 by 70 cm. each, in contact with a glass plate 3 mm. thick between them. When a continuous voltage of 2400 is applied to the condenser, the ballistic galvanometer shows a charge of 17.1 microcoulombs. What is the relative permittivity of the glass? Ans. 6.9

Prob. 3. A 0.5-mf. mica condenser is to be made out of sheets of mica 12 by 25 cm., 0.3 mm. thick, and coated on one side with a very thin film of silver. How many sheets are required? The relative permittivity of the mica is about 6.

Ans. About 96 sheets, 48 sheets in parallel per terminal.

Prob. 4. Let the dielectric in problem 2 consist, instead of glass, of three layers of different materials. Let the thicknesses of these layers be 1.2, 0.7, and 1.1 mm., and let the corresponding values of relative permittivities be 2, 3, and 5. What is the capacity of the condenser? Hint: Calculate the equivalent elastance as the sum of three elastances in series.

Ans. 2.94×10^{-3} mf.

52. Dielectric Flux Density and Electrostatic Stress (Voltage Gradient). Referring again to the uniform electrostatic field (Fig. 46), consider a cube of the dielectric, one square centimeter in cross-section, and one centimeter long in the direction of the lines of force. Let a quantity of electricity Q be supplied by the battery, as shown by the ballistic galvanometer; then the same quantity of electricity must be displaced in the dielectric. Neglecting a small displacement at the edges and at the outside surfaces of the plates, the whole quantity Q is uniformly displaced between the plates. Therefore, if the area of each plate is equal to A square centimeters, the displacement through the cube under consideration is equal to

$$D = Q/A. \quad \quad (265)$$

Since Q is the total electrostatic flux, D is naturally called the *dielectric flux density*. If Q is measured in coulombs, D is expressed in coulombs per square centimeter. In practice, Q is measured in microcoulombs, and D is expressed in microcoulombs per square centimeter. The dielectric flux density is analogous to current density U (Art. 6) and to magnetic flux density B.

When an electrostatic field is non-uniform (Fig. 48), it is conveniently subdivided by lines of force and equipotential surfaces perpendicular to the same. The procedure is similar to that used in Art. 8. In this case, the total flux or displacement divided by the area of an equipotential surface gives only the average flux density through the surface. The actual density varies from point to point, and it is therefore proper to speak of the dielectric flux density at a point. Take a tube of infinitesimal cross-section formed by lines of force, and let dQ be the displacement of electricity through this tube. The displacement is the same through any normal cross-section of the tube, because electricity behaves like an incompressible fluid. Let dA be a particular cross-section of the tube; then the flux density at this cross-section is

$$D = dQ/dA, \qquad (266)$$

D being usually expressed in coulombs (or microcoulombs) per square centimeter. Since the cross-section of the tube is infinitesimal, D is the density at the point corresponding to the position of dA.

If the flux density in a uniform field is given, the total displacement is

$$Q = DA. \qquad (267)$$

In a non-uniform field, the flux density must be given as a function of the coördinates of the field; so that

$$Q = \int_0^A D\,dA, \qquad (268)$$

the integration being extended over the whole area of an equipotential surface, or over the part of this area through which the flux is to be calculated.

The electromotive force impressed at the terminals of a condenser is balanced in the whole thickness of the dielectric; that is, each small length of path in the dielectric produces its own counter-electromotive force. Therefore, it is possible to speak of the voltage drop per unit length of the path in the

dielectric, the same as in Art. 6. This voltage gradient, or electric intensity, in a uniform field is expressed by

$$G = E/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (269)$$

and is measured, as in the conducting circuit, in volts per centimeter, kilovolts per millimeter, or in other suitable units.

In a non-uniform field, the electric intensity, or voltage gradient, varies from point to point. Let the voltage between two infinitely close equipotential surfaces MN and $M'N'$ (Fig. 48)

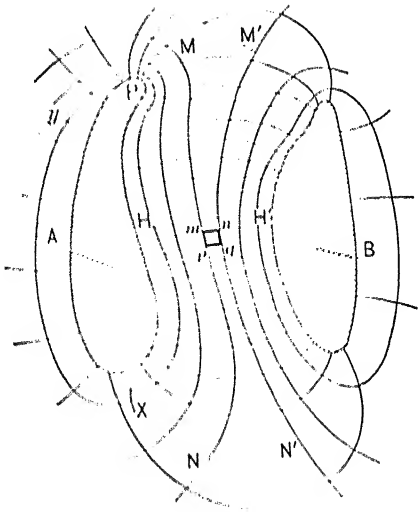

FIG. 48. A non-uniform electrostatic field, represented by lines of displacement and equipotential surfaces.

be dE, and let the distance mn between the surfaces, along a certain line of force HH', be dl. Then the voltage gradient along mn is

$$G = dE/dl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (270)$$

The length of the line mn being infinitesimal, G is the intensity at any point between m and n.

When the voltage gradient is uniform, we have for the total voltage across the field

$$E = Gl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (271)$$

In a non-uniform field, G has to be given as a function of l, so that

$$E = \int_0^l G\,dl, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (272)$$

the integration being performed between any two points on the equipotential surfaces between which the voltage is to be determined.

Imagine a uniform field existing in a dielectric, and consider a unit cube of the material. The total displacement through such a cube is equal to the flux density D, and the voltage across it is equal to the voltage gradient G. The permittance and the elastance of the cube are respectively equal to the permittivity and the elastivity of the material. Thus, applying to the cube eqs. (249) and (250), we have

$$D = \kappa G, \quad \ldots \ldots \quad (273)$$

and

$$G = \sigma D. \quad \ldots \ldots \quad (274)$$

These equations are analogous to eq. (25) in Art. 6 of this book and to eqs. (15) and (16), Art. 8, of the *Magnetic Circuit*. These relations may be considered as fundamental in the theory of the dielectric circuit, G being the cause, D the effect, and κ (or σ) the coefficient of proportionality which characterizes the material. Similar linear relations between cause and effect hold in the conduction of heat, and in the theory of elasticity. The *voltage gradient is sometimes called the stress in the dielectric*, eq. (274) being analogous to Hooke's law for elastic bodies. The elastivity σ takes the place of the modulus of elasticity.

If the field or the dielectric is non-uniform, eqs. (273) and (274) still hold true for every point, D being the dielectric flux density and G the voltage gradient at the point considered. This can be proved by applying eqs. (249) and (250) to an infinitesimal parallelopiped instead of a unit cube.

Equations (268) and (272) are expressed in words by saying that the total displacement Q is the *surface integral* of the dielectric flux density D, and the voltage E is a *line integral* of the gradient or stress G. These statements are almost self-evident from the definition of the quantities and the structure of dielectrics.

Prob. 1. What are the dielectric flux density and voltage gradient in problem 2, Art. 51? Ans. 4.885×10^{-8} mc./cm.^{2}; 8 kv./cm.

Prob. 2. The condenser specified in problem 4, Art. 51, is subjected to a difference of potential of 10 kv. What are the voltage gradients (stresses) in the three layers of dielectric?

Ans. 4.75; 3.16; 1.9 kv./mm.

CHAPTER XV

THE DIELECTRIC CIRCUIT — (*Continued*)

53. Energy in the Electrostatic Field. When a dielectric is being charged, a current flows into it from the source of electromotive force. This involves the expenditure of a certain amount of energy, because the counter-e.m.f. due to the dielectric stresses has to be overcome. This energy is not converted into heat, and lost, as in the case of metallic conduction: it is stored in the dielectric in potential form, and can be returned to the circuit by reducing the voltage at the condenser terminals. With reference to the analogy shown in Fig. 47, the mechanical energy expended by the pump in straining the elastic partition is stored in the partition, in the form of potential energy. This energy can be returned to the piston rod by allowing it to be moved by the elastic forces of the partition.

In some cases it is necessary to calculate the energy stored in an electrostatic field; or to express the energy stored per cubic centimeter of dielectric, as a function of the stress G and flux density D, at the point under consideration.

Consider first the simple case of a uniform field (Fig. 46), and neglect the small amount of displacement occurring outside the space between the plates. Let the dielectric be charged by gradually raising the voltage between its limiting surfaces from zero to a final value E; and let e and i be the instantaneous values of the voltage and charging current at a moment t during the process of charging.[1] The total electrical energy delivered to the dielectric in charging it is

$$W = \int_0^T ei\, dt = \int_0^T e \cdot dq, \quad . \quad . \quad . \quad . \quad (275)$$

where T is the total time of charging, and $dq = i\, dt$ is the infinitesimal charge or displacement added to the condenser during the interval of time dt. The quantities dq and e can be expressed

[1] The voltage and the charging current rise gradually, even though the key K be closed suddenly. This is on account of an ever-present magnetic inductance which acts as a kind of electromagnetic inertia.

through the instantaneous flux density D_t and the stress G_t; namely, from eq. (267), $dq = A\,dD_t$, and from eq. (271) $e = G_t l$. Performing the substitution, and taking the constant quantities A and l outside of the sign of integration, we get

$$W = Al \int_0^T G_t\,dD_t. \quad . \quad . \quad . \quad . \quad . \quad (276)$$

In order to integrate this expression, D_t must be expressed through G_t, or vice versa. The relation between the two is given by eq. (273). Eliminating D_t, we obtain

$$W = \kappa Al \int_0^T G_t\,dG_t = \tfrac{1}{2}\,\kappa VG^2, \quad . \quad . \quad . \quad (277)$$

where $V = Al$ is the volume of the dielectric, and G is the final value of the stress, at the time T. Hence, the energy stored per unit volume of the dielectric, or the *density of energy*, is

$$W' = W/V = \tfrac{1}{2}\,\kappa G^2 = \tfrac{1}{2}\,G^2/\sigma. \quad . \quad . \quad . \quad . \quad (278)$$

Using relations (273) and (274), the preceding formula can also be written in the following forms:

$$W' = \tfrac{1}{2}\,GD = \tfrac{1}{2}\,D^2/\kappa = \tfrac{1}{2}\,\sigma D^2. \quad . \quad . \quad . \quad . \quad (279)$$

The analogy to the corresponding formulæ in Art. 69 of the *Magnetic Circuit* is apparent at once.

The total stored energy can be expressed through the permittance or elastance of the dielectric. We have from eq. (249) $dq = C \cdot de$; substituting in eq. (275) and integrating, we get

$$W = \tfrac{1}{2}\,CE^2 = \tfrac{1}{2}\,E^2/S. \quad . \quad . \quad . \quad . \quad . \quad (280)$$

Since the final charge, or total displacement Q equals CE or E/S, the energy can be represented also in the following forms:

$$W = \tfrac{1}{2}\,QE = \tfrac{1}{2}\,Q^2/C = \tfrac{1}{2}\,Q^2S. \quad . \quad . \quad . \quad . \quad (281)$$

These formulæ are analogous to the corresponding expressions in Art. 57 of the *Magnetic Circuit*.

Let now the dielectric and the field be of an irregular form as shown in Fig. 48. The stress G and the displacement D are different at different points, so that it is necessary to consider infinitesimal layers of the dielectric between consecutive equipotential surfaces, and infinitesimal threads of displacement between the electrodes. Consider an infinitesimal volume $mnqp$ of the dielectric, comprising the part of a tube of displacement HH' between two equipotential surfaces MN and $M'N'$. The sides

mp and nq can be provided with infinitely thin metal films, because these sides lie in the equipotential surfaces, and therefore no current would flow along these metal coatings. Then the element of volume under consideration is converted into a small plate condenser; the flux density and the stress within this element can be considered as uniform, so that formula (277) holds true, and we have

$$dW = \tfrac{1}{2}\,\kappa G^2 \cdot dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad (282)$$

Differentials are used because both the volume and the stored energy are infinitesimal. The density of energy

$$W' = dW/dV = \tfrac{1}{2}\,\kappa G^2, \quad . \quad . \quad . \quad . \quad . \quad (283)$$

and has the same expression as in the case of a uniform field; but its numerical value is different from point to point, because G is variable. The other expressions for the density of energy, eqs. (278) and (279), also hold true for the points of a non-uniformly stressed dielectric, provided that proper values of D and G are used for each point.

The total energy stored in a non-uniform electrostatic field is

$$W = \tfrac{1}{2}\int_0^V \kappa G^2\, dV = \tfrac{1}{2}\int_0^V GD\, dV = \tfrac{1}{2}\int_0^V D^2\, dV/\kappa; \quad (284)$$

two more expressions may be written in which $1/\sigma$ is used in place of κ. In order to perform the integration G and D must be given as functions of coördinates, and the integration extended over the whole space occupied by the field. Equations (280) and (281) are true for condensers of any shape, because in the deduction of these formulæ no assumption is made as to the particular form of the dielectric or the electrodes.

The expressions for the electrostatic energy of the field, derived above, are analogous to the corresponding ones for the potential energy of stressed elastic bodies; and this is consistent with the assumed behavior of dielectrics. Consider the work necessary per cubic centimeter to strain mechanically the elastic fibers of a given material. The external mechanical force being applied gradually (so as to avoid oscillations), the stress varies from zero to its final value G. Let G_t be some intermediate value of the stress, and let D_t be the corresponding strain. The same symbols G and D are used here to denote the mechanical quantities analogous to electric stress and displacement. While the strain

increases from D_t to $(D_t + dD_t)$, the stress G_t may be considered constant; the infinitesimal work done is therefore equal to $G_t\,dD_t$. The total work of deformation is

$$W = \int_0^D G_t\,dD_t.$$

But, according to Hooke's law of elasticity, strains are proportional to stresses, so that a linear relation exists between D_t and G_t, similar to eq. (274). We thus arrive again at the result that the work necessary to strain one cubic unit of an elastic material is equal to $\frac{1}{2}\,\sigma D^2$.

Prob. Calculate the total stored energy, and the density of energy, in the condenser given in problem 2, Art. 51.

Ans. 20.52 milliwatt-seconds (millijoules); 19.53 microjoules per cubic centimeter.

54. The Permittance and Elastance of Irregular Paths.[1] In most practical cases where it is required to determine the permittance or elastance of a dielectric, for instance in high-tension apparatus, the geometric shapes of the metal parts and of the insulation are either irregular or too complicated to be expressed analytically. It is therefore necessary in such cases to determine the shape of the field by trials and approximations, or by experiment. The general law, substantiated by all known experiments, is as follows: *The distribution of the lines of force and equipotential surfaces in a dielectric is such as to make the total permittance a maximum, or the elastance a minimum.*

This is a particular case of the general law of nature known as the law of minimum resistance. Let a condenser of irregular shape (Fig. 48) be connected to a source of unlimited energy, having a constant voltage E. The law of minimum resistance requires that the dielectric take in as much energy as is compatible with its properties. This means that expression (280) must be a maximum; that is, with constant E, the permittance C must be a maximum, or the elastance S a minimum.

Now let it be required to establish a given flux in a certain dielectric; in other words, let Q be a constant. The law of minimum resistance requires in this case that the result be accom-

[1] The treatment is similar to that of conductors of irregular shape, given in Art. 10 of this book, and of irregular magnetic paths in Art. 41 of the *Magnetic Circuit*.

plished with the least possible expenditure of energy. According to eq. (281), we have again the same condition of maximum C or minimum S.

Therefore, in order to calculate the permittance (or the elastance) of a given dielectric, or to find the flux densities and stresses in different parts of it, proceed as follows: The field is mapped out into small cells by lines of force and equipotential surfaces, drawing them to the best of one's judgment; the total permittance is calculated by properly combining the permittances of the cells in series and in parallel. Then the assumed directions are somewhat modified, the permittance is calculated again, and so on; until by successive trials the positions of the lines of force are found with which the permittance becomes a maximum.

The work of trials is made more systematic by following a procedure suggested by Lord Rayleigh. Imagine infinitely thin sheets of metal (material of infinite permittivity) to be interposed at intervals into the field under consideration, in positions approximately coinciding with the equipotential surfaces. If these sheets exactly coincided with the actual equipotential surfaces, the total permittance of the field would not be changed, there being no tendency for the flux to pass along the equipotential surfaces. In any other position of the conducting sheets, the total permittance of the field is evidently increased. Moreover, these sheets become new equipotential surfaces of the system, because no difference of potential can be maintained along a path of infinite permittance. Thus, by drawing in the given field a system of surfaces approximately in the directions of the true equipotential surfaces, and assuming these arbitrary surfaces to be the true ones, the true elastance of the path is reduced. In other words, by calculating the elastances of the laminæ between the "incorrect" equipotential surfaces and adding these elastances in series, one obtains an elastance which is lower than the true elastance of the field. This gives a lower limit for the required elastance (or an upper limit for the permittance) of the field.

Imagine now the various tubes of force of the original field wrapped in infinitely thin sheets of a material of zero permittivity or infinite elastivity (absolute insulator). This does not change the elastance of the paths, because no flux passes between the tubes. But if these wrappings are not exactly in the direction of the lines of force, the elastance of the field is increased, because

the insulating wrappings displace the lines of force from their natural positions. Thus, by drawing in a given field a system of surfaces approximately in the directions of the lines of force, calculating the permittances of the individual tubes, and adding them in parallel, an elastance is obtained which is higher than the true elastance of the field. This gives an upper limit for the elastance (or a lower limit for the permittance) of the path under consideration.

Therefore, the practical procedure is as follows: Divide the field to the best of your judgment into cells, by equipotential surfaces and tubes of force, and calculate the elastance of the field in two ways: first, by adding the cells in parallel and the resultant laminæ in series; secondly, by adding the cells in series and the resultant tubes in parallel. The first result is lower than the second. Readjust the positions of the lines of force and the equipotential surfaces until the two results are sufficiently close to one another; an average of the last two results gives very nearly the true elastance of the field.

One difficulty in actually following out the foregoing method is that the changes in the assumed directions of the field, that will give the best result, are not always obvious. Dr. Th. Lehmann has introduced an improvement which greatly facilitates the laying out of a field.[1] While he has developed his method for the magnetic field, it is also directly applicable to the electrostatic field. We shall explain this method as applied to a two-dimensional field, though theoretically it is applicable to three-dimensional problems also. According to Lehmann, lines of force and equipotential surfaces are drawn at such distances that they inclose cells of equal elastance. Consider a slice, or a cell, in a two-dimensional field, σ centimeters thick in the third dimension, and of such a form that the average length l of the cell in the direction of the lines of force is equal to its average width w in the perpendicular direction. The elastance of such a cell is always equal to unity, no matter whether the cell itself is large or small. This follows from the fundamental formula for elastance, which in this case becomes $S = \sigma l/(\sigma \times w) = 1$.

The judgment of the eye helps to arrange cells of widths equal to their lengths, in proper positions with respect to each

[1] "Graphische Methode zur Bestimmung des Kraftlinienverlaufes in der Luft," *Elektrotechnische Zeitschrift*, Vol. 30 (1909), p. 995.

other and to the electrodes; the next approximation is apparent from the diagram, by observing the lack of equality in the average width and length of the cells. Lord Rayleigh's condition is secured automatically, since the combination of cells of equal elastance leads to the same result, whether they are combined first in parallel or in series. After a few trials the space is properly ruled, and it simply remains to count the number of cells in series and in parallel. Dr. Lehmann shows a few applications of his method to practical cases of electrical machinery, and the reader is referred to the original article for further details.

In a few simple cases, as for instance in determining the elastance between two parallel metallic cylinders of circular cross-section, or between two spheres, the principle of superposition of electric systems in equilibrium can be used, and the result obtained without trials. This principle is used in the determination of the capacity of transmission lines and cables, in the next two chapters. In two-dimensional problems, that is, in determining the shape of a field between two infinite parallel cylinders of any cross-sections whatever, the properties of conjugate functions can also be used in some simple cases; for further details see the references in Art. 10 above.

Prob. 1. Sketch empirically the field between two infinite parallel cylinders of equal circular cross-section, the distance between the centers being a few times larger than the diameter. Determine the lower and upper limits of permittance per unit of axial length, and compare the results with the theoretical formula (320) given in Art. 63 below.

Prob. 2. The terminal of a high-tension transformer consists of a long vertical rod connected to the winding, and a torus ring concentric with it, connected to the grounded case. The ring is of circular cross-section, and is placed near the center of the rod. Assuming the insulation in the whole field to be of the same permittivity, calculate by trials the elastance of the combination, with certain assumed dimensions of the rod and the ring.

55. The Law of Flux Refraction. When an electrostatic flux passes from one dielectric into another of a different permittivity (Fig. 9, Art. 11), the lines of force suddenly change their direction at the dividing surface AB between the media, and in so doing they obey the law of refraction, which is

$$\tan \theta_1 / \tan \theta_2 = \kappa_1/\kappa_2. \quad \ldots \ldots \quad (285)$$

Here θ_1 and θ_2 are the angles of incidence and refraction respectively, while κ_1 and κ_2 are the permittivities (relative or absolute)

of the two media. A similar law is proved in Art. 11 above for the electric conducting circuit, and in Art. 41a of the *Magnetic Circuit* for the magnetic flux. The proof in the case of electrostatic flux is similar in all respects to that given in Art. 11, if the student will use the words flux and flux density in place of current and current density, and permittivity in place of conductivity.

Equation (285) shows that the lower the permittivity of a dielectric the more nearly do the lines of force in it approach the direction of the normal N_1N_2 at the dividing surface. In this way the path of a displacement between two given points is shortened in the medium of lower, and lengthened in that of higher permittivity, by such an amount in each case that the total permittance of the composite condenser is larger with refraction than without it. Hence, the existence of refraction is a necessary consequence of the general law of least resistance, mentioned in the preceding article.

When mapping out an electrostatic field in two or more media, for instance, partly in a solid insulating material, partly in oil, and partly in air, the lines of force must be drawn so as to satisfy eq. (285) at the dividing surfaces. The permittance of the part of the circuit in any one of the media will not be a maximum, although the permittance of the whole combination must be a maximum. It will thus be seen that the problem, while quite simple in theory, is by no means an easy one in numerical applications, especially with the shapes of surfaces used in the construction of commercial high-tension apparatus. It is advisable for the student to train his eye in sketching lines of force in adjoining media of different permittivities, conforming the field at least approximately to eq. (285). This can be conveniently done on available drawings of high-tension transformers, switches, lightning arresters, etc.[1]

56. The Dielectric Strength of Insulating Materials. The proportionality between stress and flux density, indicated by eqs. (273) and (274), holds only up to a certain limit; in this respect it is similar to the proportionality between stresses and strains in an elastic body. After a certain limit of dielectric flux density or of voltage gradient has been exceeded, the material

[1] See also some interesting sketches and experiments in Professor W. S. Franklin's article on "Dielectric Stresses from the Mechanical Point of View," in the *General Electric Review*, Vol. 14, June, 1911.

weakens and finally breaks down. The phenomena of failure of electric insulation and the subsequent disruptive discharge are too well known to need a description here.

The values of the critical voltage gradient G_{max} and of the corresponding flux density D_{max}, at which some of the more important materials break down, are given in the last two columns of the table below. In designing insulation, the stresses must be kept well below these critical values, the factor of safety depending upon the importance of the apparatus, possibility of over-potentials, and the gradual deterioration of the insulation by heat, chemical action, moisture, and so forth. The values in the table are principally intended to give the student an idea of the order of magnitude of G_{max} and D_{max}. More accurate data will be found in electrical handbooks and pocketbooks; in important cases these design constants should be based upon test data obtained on the material in hand.

Substance.	Relative permittivity, or specific inductive capacity K.	Rupturing voltage gradient, G_{max}, in kv. per mm.	Rupturing values of dielectric flux density, D_{max}, in mc. per sq. cm.
Air	1	3	0.00265
Glass, different kinds	3–8	9–11	0.024 –0.078
Mica, natural and built up	5–8	24–40	0.110 –0.280
Porcelain	4.4	13–22	0.050 –0.085
Rubber, pure	2.2	28–35	0.055 –0.068
Rubber, vulcanized	2.7	21–28	0.050 –0.067
Transformer oil	2–2.2	14	0.025 –0.027
Vacuum	0.99	....	....

It will be seen from the second column of the table that the permittivities of solid and liquid dielectrics are larger than that of air; in other words, they are more yielding to electric stress than the air. This does not mean, however, that they break down at a lower voltage gradient than the air. On the contrary, the third and fourth columns show that the dielectrics commonly used in electrical engineering are considerably stronger electrically than the air, in that they can stand several times the electric stress and displacement at which the air breaks down.

There does not seem to be any relation between the values of elastivity and critical voltage gradient. One indicates the electrical elasticity of the material, the other its ultimate strength. They are analogous to the modulus of elasticity and the rupturing stress respectively in the mechanics of materials. Air, from an

electrical point of view, may be compared to a material of great stiffness, but one which breaks at a comparatively small elongation. On the contrary, mica may be likened to a material which is comparatively yielding, but can stand a very large elongation before it is ruptured; so that, in spite of a smaller elastivity, a much higher stress is required to rupture mica than air. The student is advised to make clear to himself these two separate properties of dielectrics. A rational design of high-tension insulation depends essentially upon a distinct understanding of them.

Dielectric strength may be properly given as the critical flux density, D_{max}, but for practical purposes it is more convenient to express it as the critical voltage gradient, G_{max}, at which the dielectric is broken down. When a dielectric is used for insulation in the form of thin sheets having a comparatively large radius of curvature, the flux density, and, consequently, the voltage gradient, are practically uniform throughout, so that $G_{max} = G_{ave}$. When, however, the layer of dielectric is thick as compared to its radius of curvature, as for instance in the insulation of high-tension machines, or when air or oil are tested between two spherical terminals, the use of the average voltage gradient $G_{ave} = E/l$ leads to wrong results. The only proper way in this case is to calculate the voltage gradient for the place where it is a maximum, and to see that it does not exceed the critical value determined from previous tests. A breakdown in one point of the dielectric results in an increase of gradient in others, and possibly in a complete failure.

Prob. 1. Show how the values in the last column of the table are derived from those in the two preceding columns.

Ans. $D_{max} = 0.08842\, KG_{max} \times 10^{-3}$.

Prob. 2. A certain material stood about 82 kv. in a layer 3.7 mm. thick. What voltage gradient can be allowed in this material at a factor of safety of 2? Ans. 11 kv. per mm.

Prob. 3. Assuming the relative permittivity of the insulation in the preceding problem to be 2.5, what is the density of energy at which the material is broken down?

Ans. 5.45×10^{-3} joules per cubic centimeter.

57. The Electrostatic Corona. The phenomena which accompany the electrical breaking-down of air deserve special mention in view of their great practical importance. When the voltage at the terminals of an air condenser is raised sufficiently

high, a pale violet light appears at the edges, at the sharp points, and in general at the protruding parts having a comparatively small radius of curvature. This silent discharge into air, due to an excessive electrostatic flux density, is called the *electrostatic corona*. In the regions where the corona appears, the air is electrically "broken down" and ionized, so that it becomes a conductor of electricity. When the voltage is raised still higher the so-called brush discharge takes place, until the whole thickness of the dielectric is broken down, and a *disruptive discharge*, or spark, jumps from one electrode to the other.

When the electrodes have projecting parts or sharp edges, the corona is formed at a voltage far below that at which the disruptive discharge occurs; the operating voltage of such devices is generally limited to that at which the corona forms. No corona is usually permissible in regular operation; first, because it may involve an appreciable loss of power; secondly, because the discharge, if allowed to play on some other insulation, will soon char and destroy it. There are cases, however, in which some corona formation is harmless. The air which is broken down becomes a part of the electrode, smoothes down the shape of the protruding metallic parts, increases their area, and thus reduces the dangerous flux density and makes it more uniform. It is of advantage to operate certain parts of a very high-tension line at nearly the critical voltage. Any voltage rise on the line due to lightning or surges is automatically relieved by a corona loss into the atmosphere; so that the line may be made self-protected, without lightning arresters.

The formation of corona must be kept in mind in the design of high-tension insulation, and in high-potential tests. Shapes and combinations of parts which lead to high or non-uniform dielectric flux densities should be avoided. Fig. 48 shows the reason why the dielectric flux density, or the potential gradient, is higher near protruding parts. The equipotential surfaces, for obvious geometrical reasons, lie closer to each other near such parts, while at a reasonable distance from the electrodes the shape of the equipotential surfaces is not affected by small irregularities in the shape of the metallic parts.

It will be seen from the table in the preceding article that the air is broken down when the voltage gradient exceeds 3000 volts per millimeter. Let this be the case at the point P (Fig. 48).

The voltage gradient has this value only at the very surface of the conductor, because the lines of force immediately spread out in the air. Thus, only a very small portion of the air is broken down and becomes part of the conducting electrode. No visual corona is formed, however. Let now the voltage be raised still further; then the next layer of air is broken down and becomes part of the electrode. When a sufficiently thick layer of air is thus ionized, a visual corona is formed around the point P. Considering the actual surface of the metal as the starting point, the voltage gradient at that point now would seem to be higher than 3000 volts per millimeter. This higher value is called the *visual* voltage gradient as distinguished from the *disruptive* voltage gradient of 3000. The student should not be misled by these names. In reality the voltage gradient does not exceed 3000, because beyond this the air becomes part of the electrode; however, the concept of visual voltage gradient is convenient in calculations.

In reality the phenomenon of ionization of air and formation of the corona is not as simple as described above, especially around conductors of small diameter, say less than 6 mm. The physical state of the layer of air adjacent to the conductor seems to be in some peculiar way affected by it, and the critical voltage gradient apparently depends in this case upon the diameter of the conductor. A discussion of numerical values and of physical theories is outside the scope of this book; and the student is referred for information to the numerous articles on the subject that appear in the leading periodicals, and in the transactions of the electrical engineering societies in this country and abroad.[1]

Quite extensive tests on corona formation, critical voltage, and the accompanying loss of power were performed by the General Electric Company, in 1910–11, and have been described by Mr. Peek.[2] The student is referred to his article for numerical data; the results are given on the first few pages of the article, and are illustrated by a numerical example.

[1] See, for instance, H. J. Ryan, "Open Atmosphere and Dry Transformer Oil as High-voltage Insulators," in the *Trans. Amer. Inst. Electr. Engrs.*, Vol. 30, Jan., 1911. This paper is a splendid exposition of the subject by one of the pioneer investigators of the corona, and contains numerous references to other articles on the subject. Professor J. B. Whitehead's experimental investigations are particularly noteworthy.

[2] F. W. Peek, Jr., "The Law of Corona and the Dielectric Strength of Air," *Trans. Amer. Inst. Electr. Engrs.*, Vol. 30, July, 1911; also Vol. 31.

Prob. 1. Assuming that under certain conditions a corona is formed when the dielectric flux density exceeds 0.0034 microcoulombs per square centimeter, calculate the factor of safety of a 25-cycle transmission line for which the charging current is 0.12 amp. per kilometer, the diameter of the conductors being 12 mm. Solution: The line is charged during 0.01 of a second, and the average charging current is $0.12/1.11 = 0.108$ amp.; hence, the maximum electrostatic displacement in the air is 1080 microcoulombs per kilometer. The surface of each conductor is 377,000 sq. cm. per kilometer, so that the density of displacement is $1080/377,000 = 0.002865$ microcoulombs per square centimeter, and the factor of safety is $34/28.65 = 1.20$.

58. Dielectric Hysteresis and Conductance. When an alternating voltage is applied at the terminals of a condenser, the dielectric is subjected to periodic stresses and displacements. If the material were perfectly elastic, no energy would be lost during one complete cycle, because the energy stored during the periods of increase in voltage would be given up to the circuit when the voltage decreased. In reality, the electric elasticity of solid and liquid dielectrics is not perfect, so that the applied voltage has to overcome some kind of molecular friction, in addition to the elastic forces. The work done against friction is converted into heat, and is lost, as far as the circuit is concerned. The phenomenon is similar to the familiar magnetic hysteresis, and is therefore called *dielectric hysteresis*. The energy lost per cycle is proportional to the square of the applied voltage, because both the displacement and the stress are proportional to the voltage.

When stresses are well below the ultimate limit of the material, the loss of power caused by dielectric hysteresis is exceedingly small. Some investigators are even in doubt as to whether it exists at all. There is often an appreciable loss of power in commerical condensers, but this loss can be mostly attributed to the fact that dielectrics are not perfect insulators. While their ohmic resistance is exceedingly high, as compared with that of metals, they nevertheless conduct some current, especially at high voltages. Thus, the observed loss of power and the heating of condensers may be simply ascribed to the I^2R loss in the insulation. Moreover, small coronas can form at the edges and projecting parts, even at the operating voltage, and thus be an additional source of loss. Some small loss is also due to the ohmic resistance and eddy currents in the metallic sheets which compose the electrodes or plates of the condenser.

An imperfect condenser, that is, one which shows a loss of power from one cause or another, can be replaced for purposes of calculation by a perfect condenser with an ohmic conductance shunted around it. This conductance, or "leakance," as some authors call it, is selected of such a value that the I^2R loss in it is equal to the loss of power from all causes in the given imperfect condenser. The actual current through the imperfect condenser is considered then as consisting of two components, — the leading reactive component through the ideal condenser, and the loss component, in phase with the voltage, through the shunted conductance. In this way, imperfect condensers can be treated graphically or analytically, according to the ordinary laws of the electric circuit.

Prob. 1. A certain kind of condenser shows a loss of power of about 17.9 watts per microfarad, at 2200 volts, 25 cycles. By what fictitious conductance should an ideal condenser be shunted, in order to replace a condenser of this kind having a capacity of 1.5 mf.?

Ans. 5.55 micromhos.

CHAPTER XVI

ELASTANCE AND PERMITTANCE OF SINGLE-PHASE CABLES AND TRANSMISSION LINES

59. The Elastance of a Single-core Cable. A cross-section of a single-core cable is shown in Fig. 49. The round conductor in the center is assumed to be solid (not stranded) for the sake of simplicity. It is surrounded by a layer of insulation, and is protected on the outside by a lead sheathing. Let such a cable be subjected to a difference of potential between the core and the sheathing; for instance, let one pole of a battery be connected to the core and the other pole to the sheathing. Let it be required to find the permittance or the elastance of the dielectric for a certain axial length l of the cable.

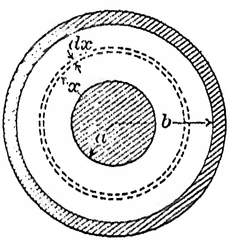

FIG. 49. A cross-section of a single-core or concentric cable.

For reasons of symmetry, the lines of force are radial straight lines between the two metal surfaces, and the equipotential surfaces are concentric cylinders. Consider the insulation to be subdivided into concentric layers of infinitesimal thickness. The elastances of these layers are all in series, so that it is sufficient to express analytically the elastance of a layer having a radius x and thickness dx, and to integrate this expression between the limits a and b, where a is the radius of the core, and b is that of the inner surface of the sheathing. The elastance of the layer in question is

$$dS = \sigma\, dx/(2\,\pi x l), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (286)$$

dx and $2\,\pi x l$ being respectively the length and cross-section of the path of the radial flux. Integrating this expression between the limits a and b gives

$$S = (\sigma/2\,\pi l)\ \mathrm{Ln}\,(b/a), \quad . \quad . \quad . \quad . \quad . \quad . \quad (287)$$

the abbreviation Ln standing for natural logarithm.

For practical calculations it is convenient to modify this formula in three respects; namely, (a) to introduce the relative permittivity K of the insulating material, (b) to express l in kilometers, and (c) to use common logarithms. Making these changes, we finally obtain

$$S = C^{-1} = (41.45/Kl) \log (b/a) \text{ megadarafs.} \quad . \quad . \quad (288)$$

For the permittance (capacity) per kilometer we have accordingly

$$C' = C/l = 0.0241\, K/\log (b/a) \text{ microfarads per kilometer.}^1 \quad (289)$$

In some cases it is necessary to know the voltage across a certain part of the insulation, for instance between the radii r and r'. Applying formula (287) to this case, for $l = 1$ cm., we get $S'_{rr} = (\sigma/2\pi) \operatorname{Ln} (r'/r)$. The voltage drop $E_{rr'}$ from r to r' is equal to this elastance multiplied by the electric displacement Q' per centimeter length of the cable. Or

$$E_{rr'} = S'_{rr'} \cdot Q' = (\sigma Q'/2\pi) \operatorname{Ln} (r'/r). \quad . \quad . \quad (290)$$

This formula finds its important application below in the calculation of the permittance of single-phase and polyphase transmission lines. It is absolutely essential to agree in regard to the signs in eq. (290). In the applications that follow, Q' is taken with the plus sign when the positive displacement is directed from the conductor, and with the minus sign when it is directed towards the conductor. It is also important to write the distances r' and r in the order given, because interchanging r' and r in eq. (290) changes the sign of $E_{rr'}$.

If the insulation consists of two or more concentric layers of different materials, the elastances of the layers are calculated separately, according to formula (288), and then added in series. The permittance of the cable as a whole is the reciprocal of this resultant elastance. The same formulæ apply to a concentric cable without sheathing, the outside conductor taking the place of the sheathing as far as stresses in the dielectric are concerned. With two cylindrical conductors side by side the elastance is calculated as shown in Art. 63 below. With three conductors the theory is rather difficult; as is also the case when the conductors are not of circular cross-section. Those interested will find

[1] This simple derivation of the formula for the capacity of a single-core cable demonstrates in a particularly striking manner the usefulness of the concept of elastance.

extensive literature on the subject in the European electrical magazines and proceedings of electrical societies. In practice, the permittance of such cables is usually determined by test.

The distribution of the electric stresses in a single-core cable is of considerable practical importance. The total displacement Q being the same through every concentric layer of the dielectric, the flux density and consequently the stress is a maximum at the surface of the inner core. For a layer of radius x we have

$$Q = D_x \cdot 2\,\pi x l = \text{const.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (291)$$

where D_x is the density of displacement through that layer. Hence,

$$D_x x = \text{const.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (292)$$

which means that the density of displacement is inversely proportional to the distance from the center. Since displacements are proportional to stresses (with a uniform insulation), we also have

$$G_x x = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (293)$$

A useful relation between the total applied voltage E and the stress G_x at a given point in the dielectric can be deduced from eq. (293). We have

$$G_x = \text{const.}/x;$$

and if we multiply both sides by dx and integrate between a and b, remembering that voltage is the line integral of intensity, we obtain

$$\int_a^b G_x dx = E = (\text{const.})\ \text{Ln}\ (b/a).$$

Eliminating the constant between these two equations, gives

$$G_x = E/[x\ \text{Ln}\ (b/a)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (294)$$

Equations (292) and (293) show that a homogeneous dielectric is fully utilized with regard to its dielectric strength only at the surface of the core, the stress gradually decreasing toward the periphery. This condition could be helped by gradually increasing the elastivity of the material toward the sheathing, so as to increase the voltage drop and the stresses there. If the elastivity of each layer could be made exactly proportional to its radius, the stress G_x would be the same throughout the dielectric. Such a condition would be an ideal one, with regard to economy in

material, provided that the dielectric strength of the "variable insulation" were constant.

This purely theoretical conclusion leads to the important practical question of the *grading of insulation* of cables. With high-tension cables, in which the thickness of insulation is large, it pays to provide two or more layers of different materials, utilizing their permittivities and ultimate strengths in the most advantageous manner. The problem is primarily to relieve the stress near the inner core, and this is done by placing near it a layer of insulation of high permittivity, so as to cause a low voltage drop. One case where the opposite arrangement would be advantageous is in a low-voltage cable in which it is desired to keep the total permittance as low as possible (for example, to obtain small capacity or low charging current at high frequencies). In this case the layer surrounding the core must have as high an elastance as possible, because it is this layer that contributes most to the total elastance of the cable. With a clear understanding of these principles, the student will be able to design a graded insulation for given conditions, if he knows the properties of the available materials.[1]

Prob. 1. A single-core cable receives a charge of 1.18 millicoulombs per kilometer when a continuous voltage of 12 kv. is applied between the core and the sheathing. The core consists of a solid conductor the diameter of which is 5 mm.; the insulation is 9.5 mm. thick. Determine the value of the relative permittivity of the material of insulation, and the extreme values of the dielectric flux density.

Ans. $K = 2.78$; $D_{max} = 0.00750$; $D_{min} = 0.00156$ microcoulombs per sq. cm.

Prob. 2. The insulation used in the cable specified in the preceding problem breaks down at a flux density of 0.062 mc. per sq. cm. Show that the critical voltage for the cable is about 70 effective a.c. kilovolts.

Prob. 3. What is the ratio between the maximum and the average stress in the insulation in problem 1? Ans. 2.42.[2]

Prob. 4. Deduce formula (290) from the fact that the voltage is the line integral of the electric intensity.

[1] For a theoretical treatment of the grading of insulation, and for the bibliography of the subject, see H. S. Osborne, *Potential Stresses in Dielectrics* (1910), a thesis presented to the Massachusetts Institute of Technology for the degree of Doctor of Engineering.

[2] There is a tendency in practice to deal with average stresses even when the field is far from being uniform. The answer to this problem shows that one has to be careful in using an average value, unless its ratio to the maximum stress is known.

Prob. 5. Show by actual calculation that in the foregoing cable the *maximum* stress in the dielectric is reduced by increasing the diameter of the conductor to 7.5 mm., with the same diameter of the sheathing. This is in spite of the fact that the insulation becomes thinner, and consequently the *average* stress greater, with the same applied voltage.

Prob. 6. Referring to the preceding problem, show that it is of advantage to make the ratio b/a about equal to ϵ, where $\epsilon = 2.71828 \ldots$ is the base of the natural system of logarithms. If the diameter of the conductor be further increased, so that the ratio b/a becomes less than ϵ, the maximum stress does not continue to decrease, but increases instead. Solution: The stress at the core is $G_a = E/[a \operatorname{Ln}(b/a)]$ according to eq. (294). As a varies, G_a reaches its minimum when $dG_a/da = 0$. Differentiating, we get

$$dG_a/da = E[1 - \operatorname{Ln}(b/a)]/[a \operatorname{Ln}(b/a)]^2 = 0;$$

whence,

$$1 - \operatorname{Ln}(b/a) = 0, \text{ or } b/a = \epsilon.$$

Prob. 7. Explain the following deduction from the theorem stated in the preceding problem. In a concentric cable subjected to an excessive voltage, if the insulation is quite thick, the layer around the inner core is first gradually destroyed or charred up to a certain thickness, and then the rest of the insulation suddenly breaks down. With a thin layer of insulation no such phenomenon is observed.

Prob. 8. A cable is provided with several concentric layers of insulation, the external radii of which are b_1, b_2, etc., and the relative permittivities, K_1, K_2, etc. Show that the elastance of the cable is expressed by the formula

$$S = (41.45/l)\,[K_1^{-1} \log(b_1/a) + K_2^{-1} \log(b_2/b_1) + K_3^{-1} \log(b_3/b_2) + \text{etc.}].$$

Prob. 9. Show that in a single-core cable the density of energy stored in the dielectric varies inversely as the square of the distance from the center.

Prob. 10. A conductor $2a$ cm. in diameter is surrounded by a concentric metal cylinder of $2b$ cm. inside diameter. What alternating voltage can be allowed between the cylinder and the conductor at a factor of safety k against the formation of corona?

Ans. $E = 18.4\,a\,(D_c \times 10^3/k) \log(b/a)$ effective kilovolts, where D_c is the flux density in microcoulombs per sq. cm., at which corona is formed.

Prob. 11. Show that the elastance of the dielectric between two concentric spheres of radii a and b is equal to $(\sigma_a/4\pi K)(1/a - 1/b)$ megadarafs.

Prob. 12. Show that with two concentric spheres the equation corresponding to (294) is $G_x = E/[x^2(a^{-1} - b^{-1})]$.

Prob. 13. Apply the formulæ given in the text above to the theory of a condenser-type terminal.[1]

[1] See A. B. Reynders, "Condenser Type of Insulation for High-tension Terminals," *Trans. Amer. Inst. Electr. Engrs.*, Vol. 28 (1909), p. 209.

60. The Elastance of a Single-phase Line. The general character of the electrostatic field between two infinite parallel conductors is shown in Fig. 50. The lines of force are arcs of circles extending from one metal surface to the other; the equipotential surfaces are circular cylinders eccentric with respect to the conductors (see Art. 62 below). It is required to calculate the elastance of the air between the two conductors, for a unit axial length of the line. Knowing this elastance, the charging current of the line can be calculated for a given frequency. This elastance, or its reciprocal, the permittance, is used in the predetermination of the regulation of a transmission line (Art. 68 below).

We shall consider in this article the usual practical case in which the radius a of the conductors is small as compared with the interaxial distance b. It is shown in Art. 63 below how to determine the elastance when the diameters of the cylinders are comparatively large.

For purposes of analysis it is convenient to consider the field shown in Fig. 50 as the result of the superposition of two simple radial fields similar to that in Fig. 49. Consider the conductor A, together with a concentric cylinder of an infinitely large radius, as one electric system. Let the conductor B with a similar concentric cylinder form another independent system. Let the conductor A be connected to the positive pole of a battery of voltage E, the conductor B to the negative pole, and the two cylinders at infinity to the middle point of the battery. In the first concentric condenser the displacement of positive electricity is from the conductor A to the infinite cylinder, while in the second system the positive displacement is from the infinite cylinder toward the conductor B. The displacements due to the two systems are equal and opposite at the two infinite cylinders, and the cylinders themselves coincide at infinity, because the distance AB between their axes is infinitely small as compared with their radii. Hence, the two displacements at the cylinders cancel each other, and the combination of the two cylindrical condensers is electrically identical with the two given parallel conductors A and B.

In a medium of constant permittivity the resultant stress or voltage gradient, produced at a point by the combined action of two or more independent electric systems, is equal to the geometric sum of the stresses produced at the same point by each system.

This *principle of superposition* can be considered either as an experimental fact or as an immediate consequence of the fact that in a medium of constant permittivity the effects are proportional to the causes. This principle being true for electric intensities, the component flux densities at a point are also combined according to the parallelogram law, because they are proportional to the intensities. Hence, the resultant electrostatic flux can be regarded as the result of the superposition of the fluxes created by

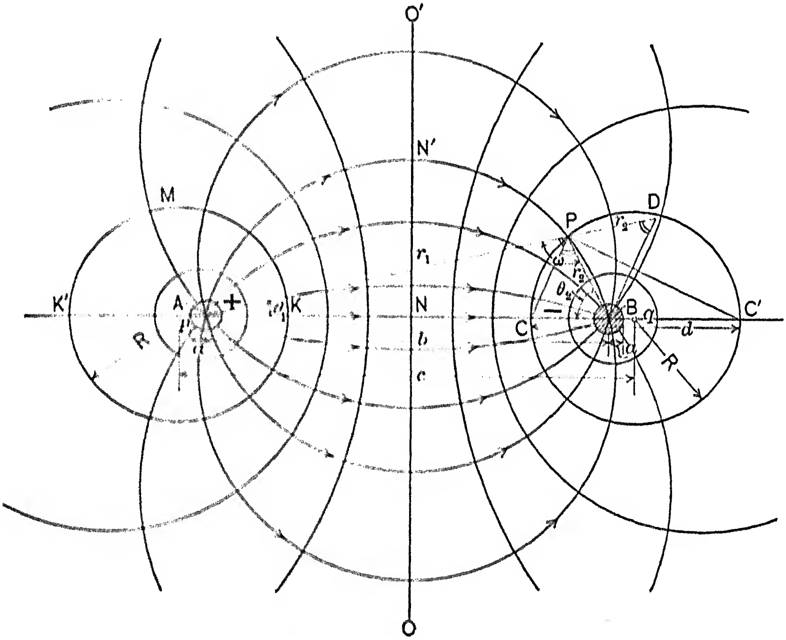

FIG. 50. The electrostatic field produced by a single-phase transmission line.

the component systems. Furthermore, the actual voltage between any two points in the dielectric is the algebraic sum of the voltages due to the component systems, because each voltage is the line integral of the corresponding voltage gradient, and the principle of superposition is valid for these gradients. This line integral is a function only of the positions of the two points, and is independent of the path along which the integration is performed. This latter fact is very convenient in applications of the principle to the solution of problems.

In order to be able to apply the formulæ deduced in the preceding article, it is essential that the diameters of the two wires be small as compared with the distance between them. The reason is that each component system is supposed to possess a radial field, in spite of the presence of the other conductor. This is practically true when the second conductor is so small, or so far distant from the first, that the infinite permittivity of its material does not appreciably distort the radial field. To be more precise, the distortion of the radial component field originating from each conductor, caused by the presence of the other, must be negligible.

It is sufficient to calculate the elastance of that part of the system between one of the conductors and the neutral plane of symmetry OO', the total elastance being equal to twice that value. This we can do by computing the voltage needed to produce a displacement Q' per unit length of the line. The voltage between the surface of the conductor A and the point N in the plane of symmetry OO' is equal to $\frac{1}{2} E$. On the other hand, the same voltage can be expressed as the sum of the voltages due to the two component systems. Referring to eq. (290), let the distance r' refer to the point N, and let r refer to a point on the surface of the conductor A. Then, as far as the first component system is concerned, the voltage between A and N is equal to $(\sigma Q'/2\pi) \text{ Ln } (\frac{1}{2} b/a)$, where Q' is the actual displacement per unit length of the line. In the second system, the voltage between the same two points is $-(\sigma Q'/2\pi) \text{ Ln } (\frac{1}{2} b/b)$. The minus sign is due to the fact that the displacement in the second system is toward the conductor B, and hence must be considered as negative if that at the first conductor is regarded as positive. The ratio r'/r for the second system is more accurately equal to $\frac{1}{2} b/(b - a)$, but a, being by supposition small as compared to b, is neglected in the denominator. Equating the sum of the preceding two expressions for the voltage between A and N to the actual voltage $\frac{1}{2} E$, we get

$$\tfrac{1}{2} E = (\sigma Q'/2\pi) \text{ Ln } (b/a). \quad . \quad . \quad . \quad . \quad (295)$$

Hence, the elastance between one of the conductors and the neutral plane OO', for a unit of axial length, is

$$S' = (C')^{-1} = \tfrac{1}{2} E/Q' = (\sigma/2\pi) \text{ Ln}(b/a), \quad . \quad (296)$$

or, with air as the dielectric,

$$S' = (C')^{-1} = 41.45 \log (b/a) \text{ megadarafs per kilometer.} \quad (297)$$

The corresponding permittance is

$$C' = (S')^{-1} = 0.0241/\log (b/a) \text{ microfarads per kilometer.} \quad (298)$$

When using these formulæ, one must not forget that the permittance is proportional to the length of the line, while the elastance varies inversely as the length of the line. The total elastance for a unit length between the two conductors is equal to $2\ S'$, the corresponding permittance being $\frac{1}{2}\ C'$.

Prob. 1. For a few standard spacings and sizes of conductor, check the values of permittance given by eq. (298) with those tabulated in an electrical pocketbook.

Prob. 2. For some assumed values of a, b, and E, corresponding to an actual transmission line, plot a curve of values of the voltage gradient along the line AB, and also draw the horizontal straight line representing the average gradient E/b.

Hint: At a distance x from A the intensity due to the system A is $\sigma Q'/(2\ \pi x)$; that due to the system B is $\sigma Q'/[2\ \pi\ (b - x)]$, both intensities being directed from left to right.

Prob. 3. In Fig. 59 let A and B be small spheres, instead of cylinders. Show that the elastance between one of the spheres and the neutral plane OO' is equal to $(\sigma/4\ \pi)\ (1/a - 1/b)$. Hint: Apply the principle of superposition, as in the text above, and utilize the solutions of problems 11 and 12 of the preceding article.

Prob. 4. In a transmission line the wire B is split into two separate conductors B_1 and B_2, connected in parallel. The spacings $A - B_1$, $A - B_2$, and $B_1 - B_2$ are equal to b_1, b_2, and b_{12} respectively. Show how to calculate the total permittance per unit length of the line, using the method of superposition. Solution: Let Q', Q_1', and Q_2' be the displacements issuing per unit length of the conductors A, B_1, and B_2 respectively. Resolve the given system into three systems, with the three given conductors each concentric with a cylinder of infinite radius. Then we have the following three conditions: (a) $Q' + Q_1' + Q_2' = 0$, because electricity behaves as an incompressible fluid; (b) the given voltage E between A and B_1 is the sum of the partial voltages for the three component systems, each expressed according to eq. (290); and (c) the same is true for the voltage E between A and B_2.[1] From these three equations the quantities Q_1' and Q_2' are eliminated, and the required elastance is determined from the resultant equation, as the ratio of E to Q'.

Prob. 5. Show how to calculate the elastance between two small cylinders or spheres of unequal radii.

Prob. 6. Analyze the formal mathematical reason for which the electrostatic equipotential lines in Figs. 49 and 50 coincide with the magnetic lines of force, and vice versa. Compare Figs. 46 and 47, Arts. 59 and 60, in the *Magnetic Circuit*.

[1] Or else we may use as condition (c) the fact that the resultant voltage between B_1 and B_2 equals zero.

61.[1] **The Influence of the Ground upon the Elastance of a Single-phase Line.** When the ground is used as the return conductor of a circuit, for instance in single-phase railways and in telegraph lines, the elastance of the circuit is calculated by assuming the ground to be a good conductor of electricity; in other words, its permittivity is assumed to be infinitely large. This gives a larger permittance than any other assumption, and consequently a value which is on the safe side. According to the law of refraction (Art. 55) the lines of force from the metallic conductor enter the ground at right angles to its surface; so that the field has the shape shown in Fig. 50, between one of the wires and the plane of symmetry OO', which in this case represents the surface of the ground. This leads to Lord Kelvin's *method of electric images*, which we shall use in its simplest form only.

When it is required to find the shape of the field, or the elastance between a conductor such as A and an infinite conducting surface such as OO', first locate a fictitious conductor B, which is the electric image of the conductor A; that is, B is located as if it were the optical image of A in the plane mirror OO'. Furthermore, if A has a potential of E volts above that of OO', take the potential of B as E volts below that of OO', the voltage between A and B thus being $2E$ volts. Having located B, the conducting plane OO' is removed, and the field between A and B is determined. The part of the field between A and OO' has real existence, that between OO' and B is fictitious.

The validity of this principle in the case under consideration becomes evident by the following reasoning: Let a voltage of $2E$ be maintained between A and B by means of a battery. Place an infinite conducting sheet of negligible thickness so as to coincide with the equipotential plane OO'. The field is not affected thereby, the lines of displacement being normal to this sheet. Connecting the sheet to the middle point of the battery does not in any way disturb the field. Now the field between A and the sheet OO' is maintained by one half of the battery, that between B and OO' by the other half. Both halves are in equilibrium independently of each other, so that the conductor B with its half of the battery may be removed without disturbing the field between A and OO'. Conversely, to find the field between

[1] The rest of this chapter may be omitted if so desired, as it is not necessary to an understanding of the remainder of the book.

A and a conducting sheet OO', the latter is replaced by a fictitious conductor B, so as to reduce the conditions to those investigated in the previous article. For a general discussion of the principle of electric images, see any standard work on the mathematical theory of electricity and magnetism.

Applying this principle to the case of a single-phase line with ground return, we see immediately that all of the formulæ deduced in the preceding article hold true, provided that we put $b = 2h$, where h is the elevation of the conductor above the ground. Since this elevation is usually quite considerable, it will be seen that the elastance of the circuit is larger, and the charging current smaller as compared to the case of a metallic circuit having a comparatively small spacing.

The next case to be considered is that of the elastance of a metallic return line, as reduced by the proximity of the earth (Fig. 51). The elastance is reduced as compared to that in Fig. 50 because part of the medium of finite elastivity (air) is replaced by the ground, which is assumed to be of zero elastivity, or a good conductor of electricity. It will be seen from the figure that the lines of force are deflected toward the ground, where they find a path of less elastance.

The total elastance between the conductors A and B is calculated, using again the method of electric images. The lines of force meet the ground at right angles, and its surface is one of equal potential. The field above the ground would be the same if the ground were removed and replaced by the electric images A' and B' of the wires, the polarity of the images being that indicated in the sketch. The surface of the ground becomes now a plane of symmetry. The fictitious field below the ground is indicated by the dotted lines. The dielectric field may now be considered as if due to a superposition of four systems, each consisting of one of the conductors and a cylinder at infinity. Applying eq. (290) for each of these four systems, we find that the voltage between A and B,

due to system A, is $+(\sigma Q'/2\pi)\,\text{Ln}\,(b/a)$;
due to system B, is $-(\sigma Q'/2\pi)\,\text{Ln}\,(a/b)$;
due to system A', is $-(\sigma Q'/2\pi)\,\text{Ln}\,(2d/2h_1)$;
due to system B', is $+(\sigma Q'/2\pi)\,\text{Ln}\,(2h_2/2d)$;

where $2d = AB' = A'B$. The actual voltage between A and B

being equal to E, we have, by adding the preceding four expressions,

$$E = (\sigma Q'/2\pi)\,[2\,\mathrm{Ln}\,(b/a) - \mathrm{Ln}\,(d^2/h_1h_2)]. \quad . \quad . \qquad (299)$$

One half of the elastance between A and B is

$$S' = \tfrac{1}{2}\,E/Q' = (\sigma/2\pi)\,[\mathrm{Ln}\,(b/a) - \tfrac{1}{2}\,\mathrm{Ln}\,(d^2/h_1h_2)]. \qquad (300)$$

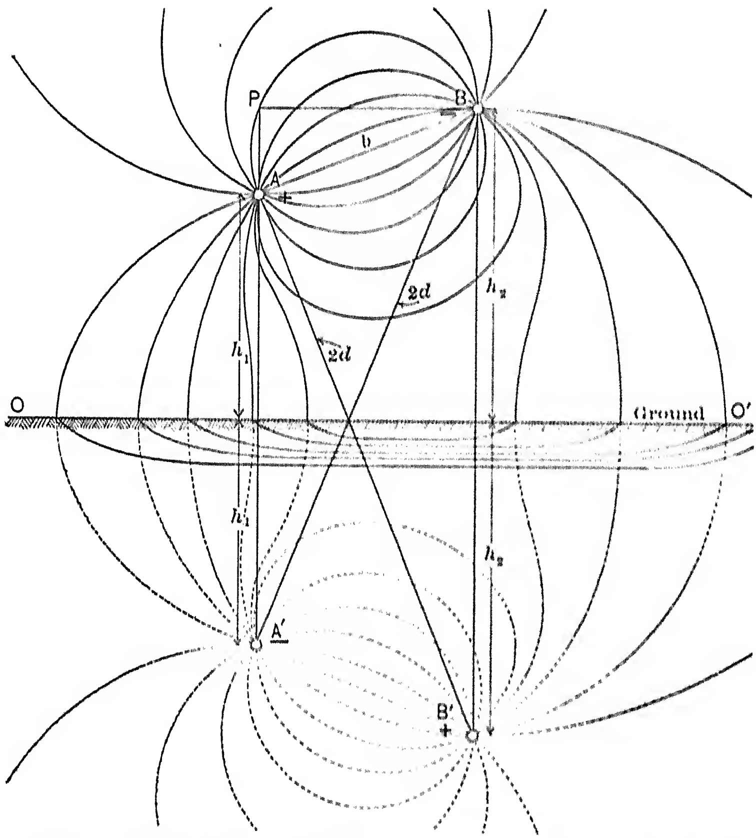

FIG. 51. The electrostatic field due to a single-phase line AB, as affected by the proximity of the ground. A' and B' are the electric images of A and B.

This expression is identical with formula (296), except for the last term, which represents the reduction in elastance due to the presence of the ground. When the distances h_1 and h_2 to the ground are large as compared to the spacing b, the ratio d^2/h_1h_2 differs but little from unity, and the correction is small because the logarithm

of unity is equal to zero. Equation (300) can be written also in a simpler form by combining the two logarithms into one. We obtain then

$$S' = (\sigma/2\,\pi)\ \mathrm{Ln}\,(b_c/a), \quad . \quad . \quad . \quad . \quad . \quad (301)$$

where b_c stands for the *corrected* spacing, determined from the expression

$$b_c = b\,(\sqrt{h_1h_2}/d). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (302)$$

We have thus arrived at the following simple rule: *The elastance and permittance of a single-phase line, with the effect of the ground considered, are expressed by the same formulæ* (296) *to* (298) *as though this effect were ignored, provided that the actual spacing b is replaced by the corrected spacing b_c given by formula* (302) *or* (305).

In practice, the values of h_1, h_2, and b are known, and it is desirable to avoid the use of the quantity d in the foregoing formula. Applying a well-known theorem of elementary geometry, we have from the triangle $AA'B$

$$\overline{A'B}^2 = \overline{AA'}^2 + \overline{AB}^2 + 2\,\overline{AA'} \times \overline{AP},$$

or

$$4\,d^2 = 4\,h_1^2 + b^2 + 4\,h_1(h_2 - h_1),$$

from which

$$4\,d^2 = b^2 + 4\,h_1h_2.$$

Hence

$$d^2/h_1h_2 = 1 + \tfrac{1}{4}\,b^2/h_1h_2. \quad . \quad . \quad . \quad . \quad . \quad (303)$$

Equation (300) becomes then

$$S' = (\sigma/2\,\pi)\,[\mathrm{Ln}\,(b/a) - \tfrac{1}{2}\,\mathrm{Ln}\,(1 + \tfrac{1}{4}\,b^2/h_1h_2)], \quad . \quad (304)$$

and from eq. (302) we have

$$b_c = b/\sqrt{1 + \tfrac{1}{4}\,b^2/h_1h_2}. \quad . \quad . \quad . \quad . \quad . \quad (305)$$

When tables of capacity for standard spacings are used, as tabulated in various reference books, the correction for the influence of the ground will be found convenient in the form shown in problem 2 below.

Prob. 1. For various usual spacings and sizes of conductor, calculate the per cent error introduced in computing the permittance of a transmission line by neglecting the influence of the earth in the most unfavorable cases. Select the conductors either in a vertical or in a horizontal plane, whichever arrangement in your opinion is more affected by the ground.

Prob. 2. When permittances are taken from standard tables, it is not convenient to use the corrected spacing b_c, because capacities are tabulated for standard spacings only. In this case it is convenient to represent the elastance given by eq. (304) in the form $S_c' = S' - s$, where S' is the reciprocal of the value of capacity found in the tables, and s is the correction due to the presence of the ground. Deduce a simple form of this correction, when it is small. Solution: Expanding the natural logarithm according to the series $\mathrm{Ln}\,(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \text{etc.}$, we find that the correction $s = 9\,\mathrm{Ln}\,(1 + \frac{1}{4}b^2/h_1h_2) = 9\,[\frac{1}{4}b^2/h_1h_2 - \frac{1}{2}(\frac{1}{4}b^2/h_1h_2)^2 + \text{etc.}]$ in megadarafs per kilometer of one conductor. S' must of course be taken also in megadarafs per kilometer of one conductor.

Prob. 3. Deduce formulæ for the influence of the ground in the case of small spheres in place of the cylinders.

62. The Equations of the Electrostatic Lines of Force and Equipotential Surfaces Produced by a Single-phase Line. In Fig. 50, let P be a point on the line of force $AN'PB$, of which we desire to find the equation. Let us calculate the total flux which passes from conductor A to B between the plane of symmetry AB, and the surface of force on which the point P is located.[1]

Let the axial length for which the flux is determined be equal to one centimeter. This flux may be considered as the resultant of the fluxes due to the systems A and B. The radial flux passing between AB and P, due to the component system having the center A, is equal to $Q'\theta_1/2\,\pi$, and is directed from left to right. The flux due to the B system is equal to $Q'\theta_2/2\,\pi$, and is also directed from left to right, B being the negative conductor. The total or the actual flux between the surfaces of force AB and $AN'PB$ equals $(Q'/2\,\pi)\,(\theta_1 + \theta_2)$. Since this flux does not depend upon the position of the point P, provided that the point is taken upon the line of force under consideration, we have

$$\theta_1 + \theta_2 = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (306)$$

for all points on a line of force. This is the equation of the line of force. For different lines of force, the value of the constant is different. In the triangle APB, the angle ω is supplementary to

[1] It is convenient to speak of a surface formed by lines of force as a *surface of force.* For instance, a line of force such as $AN'PB$, should it move in a direction perpendicular to the plane of the paper, would form a cylindrical surface, which we shall call a surface of force, by analogy with a line of force. On the other hand, we shall call a line such as CPC' an *equipotential line*, to distinguish it from the corresponding equipotential surface which it represents in the sketch.

the sum of the angles θ_1 and θ_2, so that condition (306) may also be written

$$\omega = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (307)$$

This represents the arc of a circle passing through A and B, of which ω is the inscribed angle. It is thus proved that the lines of force are arcs of circles passing through A and B.

For points on the line of symmetry OO' the angles θ_1 and θ_2 are equal, so that the total flux corresponding to a certain angle θ_1 is $(Q'/2\pi)(2\theta_1) = Q'\theta_1/\pi$. This fact permits us to mark on the line OO' the intersections of the surfaces of force between which are included definite fractions of the total flux Q'. For instance, let it be desired to draw a line of force such that the flux between it and the plane AB shall equal one sixth of the total flux. One sixth of 180 degrees are 30 degrees; we therefore draw from A a straight line at an angle of 30 degrees to AB, and through its intersection with OO' draw an arc of a circle passing through A and B. In this way, the total flux, or what is the same, the total permittance between A and B, can be divided into any number of equal or unequal permittances in parallel.

To prove that the equipotential lines are also circles, take again a point P determined by the distances r_1 and r_2 from A and B respectively. If the point C lies on the same equipotential line, the voltage between P and C is equal to zero, so that, applying eq. (290) for the two component systems, we get

$$(\sigma Q'/2\pi)\,\text{Ln}\,(r_1/AC) - (\sigma Q'/2\pi)\,\text{Ln}\,(r_2/BC) = 0,$$

from which

$$\text{Ln}\,(r_1/AC) = \text{Ln}\,(r_2/BC),$$

or

$$r_1/r_2 = AC/BC = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (308)$$

This is the equation of an equipotential line in "bipolar" coördinates; the curve is such that the ratio of r_1 to r_2 remains constant. This constant is different for each equipotential line, because each line has its own point C.

Equation (308) may be proved to represent a circle, by selecting an origin, say at A, and substituting for r_1 and r_2 their values in terms of the rectangular coördinates x and y. The following proof by elementary geometry leads to the same result. Produce AP and lay off $PD = PB = r_2$. According to eq. (308), BD is parallel to CP, and consequently PC bisects the angle $APB = \omega$.

Let the point C' lie on the same equipotential line with C; then the voltage between P and C' is also equal to zero, and by analogy with eq. (308) we have

$$r_1/r_2 = AC'/BC' = \text{const.} \quad . \quad . \quad . \quad . \quad (309)$$

By plotting $PD' = r_2$ (not shown in the figure) along PA, in the opposite direction from PD, and connecting D' to B, one can show as before that PC' bisects the angle $BPD = 180° - \omega$. But the bisectors of two supplementary angles are perpendicular to each other; consequently, CPC' is a right angle, and the point P lies on a semicircle drawn on the diameter CC'. This semicircle is the equipotential line itself, because all points, such as P, which are determined by C and C', must lie on it. The semicircle below AB evidently belongs to the same equipotential line.

From eqs. (308) and (309) the following expressions are obtained for the radius R of the equipotential line under consideration.

$$R = \frac{BC}{1 - (BC/AC)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (310)$$

or

$$R = \frac{BC'}{1 + BC'/AC'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (311)$$

so that the equipotential line can be easily drawn for a given C or C'.

Let it be required to calculate the elastance of the slice of dielectric between the neutral plane OO' and the equipotential surface passing through a given point C. It is sufficient to find the expression for a unit axial length, knowing that the elastance is inversely proportional to the length of the conductors.

Write the expression for the voltage between the points N and C, using again eq. (290). For the systems A and B we have

$$E_{NC} = (\sigma Q'/2\pi)\,[\text{Ln}\,(AC/AN) - \text{Ln}\,(BC/BN)],$$

or, since $AN = BN$,

$$E_{NC} = (\sigma Q'/2\pi)\,\text{Ln}\,(AC/BC). \quad . \quad . \quad . \quad . \quad (312)$$

From this equation we see that the elastance per centimeter

$$S_{NC}' = E_{NC}/Q' = (\sigma/2\pi)\,\text{Ln}\,(AC/BC). \quad . \quad . \quad (313)$$

From this expression, the elastance between any two equipotential surfaces can be calculated, by computing first the elastance be-

tween each of the surfaces and the plane of symmetry OO', and then taking either the sum or the difference of these elastances, depending upon the positions of the two given surfaces; that is, whether they lie on different sides or on the same side of the plane OO'.

It has been explained above how to divide the field by surfaces of force into permittances of desired values, these permittances being proportional to the angles θ_1 or θ_2, determined by the point N' on the neutral plane. Knowing now how to subdivide the field into elastances of desired values by equipotential surfaces, the student can without difficulty calculate the permittance or the elastance of a given slice in the field between two equipotential surfaces and two surfaces of force.

Prob. 1. For an assumed size of the conductors and a spacing used in extra-high-tension transmission lines, draw a set of lines of force and equipotential lines (Fig. 50) such as to divide the total voltage and the total electrostatic flux into 10 equal parts.

Prob. 2. Let A and B (Fig. 50) be two very small wires at a distance of 90 cm. between centers. What is the permittance of the slice $NN'PC$ if $NN' = 25$ cm.; $NC' = 32$ cm.; and the axial length is 180 m.?

Ans. 0.000861 mf.

Prob. 3. Show that the lines of force between two small spheres are not circles, but curves the equation of which is $\cos\theta_1 + \cos\theta_2 = \text{const.}$

Prob. 4. Show that the equipotential surfaces in the case of two small spheres are represented by the equation $1/r_1 - 1/r_2 = \text{const.}$

Prob. 5. Show how to draw in a given case the field shown in Fig. 51. Solution: Draw a set of n lines of force due to the system AB alone, the same as in Fig. 50. Let the flux Q' be divided into n equal parts, so that the flux between the adjacent surfaces of force is Q'/n. Draw a similar set of lines of force for the system $A'B'$. The equation of a line of force in the system AB is $\omega = \text{const.}$; that in the system $A'B'$ is $\omega' = \text{const.}$ According to the principle of superposition, the equation of a line of force in the resultant field is $\omega - \omega' = \text{const.}$, the minus sign being due to the fact that A' is negative if A is positive. Let the point of intersection of two lines of force $\omega = C'$ and $\omega' = C''$ be the starting point for drawing a line of force in the resultant field. Then the point of intersection of the next lines $\omega = C' + \pi/n$ and $\omega' = C'' + \pi/n$ also belongs to the same line of force in the resultant field, because for both points $\omega - \omega' = C' - C''$. In other words, the lines of force in the resultant field are *diagonal curves* with respect to the lines of force in the component fields, and may be drawn from intersection to intersection. A similar construction holds for equipotential surfaces. The student is strongly urged to try this construction for some assumed data, because the method of diagonal curves is generally applicable when a given field can be resolved into two simpler fields.

63. The Elastance between Two Large Parallel Circular Cylinders. The formulæ derived in Art. 60, for the elastance and permittance of a homogeneous medium between two parallel cylinders, hold true only when the diameters of the cylinders are small as compared to the interaxial distance, for the reason there explained. When the diameters of the cylinders are comparatively large, the elastance is derived by reducing the conditions to those obtaining in Art. 60.

Let A and B (Fig. 50) represent as before two conductors of very small diameter, and let a difference of potential of 100 volts be maintained between them by means of a battery. Let the voltage between the conductor B and the equipotential surface CPC'' be 20 volts. Place an infinitely thin metal sheet so as to coincide with this surface, and connect this sheet to a point of the battery such that the voltage between it and the conductor B still remains equal to 20 volts. These changes do not affect the electrostatic field either inside or outside the surface CPC'', the displacement being normal to this surface. Now remove the conductor B altogether, leaving a difference of potential of 80 volts maintained by the battery between the conductor A and the cylinder CPC''. The field outside the cylinder is not affected; that inside of it has entirely disappeared. We have now a field between the cylinder A of very small diameter and the cylinder CPC'' of a comparatively large diameter. Take now another equipotential surface, for instance KMK', symmetrical with CPC'', place a metal cylinder so as to coincide with it, and connect it to a tap on the battery, so that the same difference of potential of 20 volts remains between this cylinder and the conductor A. The field is not altered by this connection, and now the conductor A may be removed. Thus, we finally obtain a field between two cylinders of comparatively large diameter. The difference of potential between the cylinders is only 60 volts, while the original difference of potential between the conductors A and B was 100 volts.

Conversely, let the cylinders CPC'' and KMK' be given, and let it be required to find the shape of the field between them, and the elastance of this field. The problem is reduced to that of finding the positions of the infinitely small eccentric conductors A and B, with respect to which the given cylinders are equipotential surfaces. Then the field is mapped out according to the formulæ given in the preceding article, leaving out the space

inside the cylinders. The elastance between one of the large cylinders and the plane OO' is calculated by using formula (313). This method is applicable whether the two cylinders are of the same radius or not, and whether one is outside or inside of the other. It is always possible to find the positions of the lines A and B with respect to which the given cylinders represent equipotential surfaces. The details of the calculation are given below.

Consider first the case of two cylinders CPC' and KMK' of the same diameter d; let the distance between the centers p and q of these cylinders be equal to c. In order to use eq. (313), it is necessary to express AC and BC through the given quantities c and d. According to eqs. (308) and (309), we have

$$AC/BC = AC'/BC'. \quad . \quad . \quad . \quad . \quad . \quad (314)$$

All the quantities which enter into this equation can be expressed through one unknown length, for instance BC. We put

$$BC = AK = x;$$

then

$$\left.\begin{aligned} AC &= CK + AK = (c - d) + x; \\ AC' &= x + c; \\ BC' &= d - x. \end{aligned}\right\} \quad . \quad . \quad . \quad (315)$$

Substituting these values into eq. (314), and solving the resulting quadratic equation for x, we obtain, retaining the positive value only,

$$\begin{aligned} BC = AK = x &= \tfrac{1}{2}[-(c-d) + \sqrt{c^2 - d^2}] \\ &= \tfrac{1}{2}d[-(\alpha - 1) + \sqrt{\alpha^2 - 1}], \quad . \quad . \quad . \quad . \quad . \quad (316) \end{aligned}$$

where the ratio of the interaxial distance to the diameter is denoted by α, or

$$\alpha = c/d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (317)$$

By substituting this value of x into the expression for AC in eqs. (315), we find

$$AC = \tfrac{1}{2}[(c-d) + \sqrt{c^2 - d^2}] = \tfrac{1}{2}d[(\alpha - 1) + \sqrt{\alpha^2 - 1}], \quad (318)$$

so that

$$AC/BC = [(\alpha - 1) + \sqrt{\alpha^2 - 1}]/[-(\alpha - 1) + \sqrt{\alpha^2 - 1}].$$

This expression can be simplified by multiplying both the numerator and the denominator by the value of the numerator, so

as to get rid of the square root in the denominator. The result is

$$AC/BC = \alpha + \sqrt{\alpha^2 - 1}. \quad . \quad . \quad . \quad . \quad (319)$$

The expression (313) for the elastance between one of the cylinders and the plane of symmetry, per unit of axial length, becomes

$$S' = (\sigma/2\pi) \operatorname{Ln} [\alpha + \sqrt{\alpha^2 - 1}]. \quad . \quad . \quad . \quad (320)$$

Those familiar with hyperbolic functions will notice that the preceding equation can be simplified into

$$S' = (\sigma/2\pi) \operatorname{Cosh}^{-1} \alpha. \quad . \quad . \quad . \quad . \quad . \quad (321)$$

Since tables of hyperbolic functions are readily available, the evaluation of elastance is simpler in this form than it is if eq. (320) is used.[1]

When the diameter of the conductors is small as compared to the interaxial distance, α is a large quantity, and unity under the radical sign in eq. (320) may be neglected. This equation becomes then practically identical with eq. (296). For large values of α, the term $(1-1/\alpha^2)^{\frac{1}{2}}$, obtained by factoring in expression (320), is conveniently expanded according to the binomial theorem, the result being

$$S' = (\sigma/2\pi) \operatorname{Ln} (2\alpha - \tfrac{1}{2}\alpha^{-1} - \tfrac{1}{8}\alpha^{-3} - \tfrac{1}{16}\alpha^{-5} - \ldots). \quad (322)$$

With the exception of 2α, all of the terms in parentheses are small corrections to the result.

Let now the diameters of the two given cylinders be different. In addition to relation (314), we also have

$$BK/AK = BK'/AK'. \quad . \quad . \quad . \quad . \quad . \quad (323)$$

It is necessary in this case to introduce two unknown quantities, $BC = x$ and $AK = y$. Equations (315) are modified accordingly. All of the quantities in eqs. (314) and (323) are expressed through x and y, and then these two equations are solved together for x and y. After this, the elastance between each cylinder and the plane OO' is expressed by using eq. (313).

[1] Dr. A. E. Kennelly, "The Linear Resistance between Parallel Conducting Cylinders in a Medium of Uniform Conductivity," *Proceedings Amer. Philosophical Soc.*, Vol. 48 (1909), p. 142; also his article on "Graphic Representations of the Linear Electrostatic Capacity between Equal Parallel Wires," *Electrical World*, Vol. 56 (1910), p. 1000. See also his book on *Applications of Hyperbolic Functions to Electrical Engineering* (1912).

In some cases it is required to calculate the *dielectric flux density* at a point in the field between the cylinders, or at the surface of one of the cylinders. Let P (Fig. 52) be a point in the field between two parallel cylinders, small or large; the flux density at P is the geometric sum of the densities due to the systems A and B. The flux density due to the system A is

$$D_1 = Q'/(2\pi r_1),$$

while that due to B is

$$D_2 = Q'/(2\pi r_2).$$

These component densities are directed as shown in Fig. 52. The resultant density D is directed along the tangent to the line

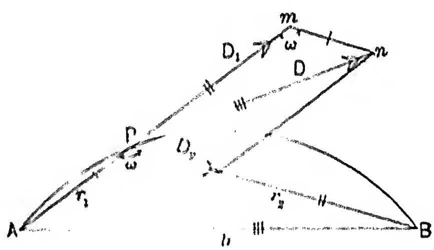

FIG. 52. Dielectric flux density at a point, determined by the method of superposition.

of force through P. From the preceding two equations, we have the relation

$$D_1 : D_2 = r_2 : r_1,$$

so that the triangles APB and Pmn are similar. The corresponding sides are marked with one, two, and three short lines respectively. From these triangles we can write

$$D : D_1 = b : r_2,$$

or, substituting the foregoing expression for D_1,

$$D = Q'b/(2\pi r_1 r_2). \quad \ldots \ldots \quad (324)$$

From this expression, the flux density can be calculated at any point in the field or on the surface of one of the cylinders. Multiplying the flux density by the elastivity of the medium, the corresponding dielectric stress is obtained. It must be kept well in mind that b, r_1, and r_2 refer to the points A and B, and not to the centers p and q of the actual cylinders.

Prob. 1. Take two equal cylinders at a comparatively short distance apart, and (a) calculate the permittance per meter of the axial length; (b) divide the field into 10 equal elastances in series and into 10 equal permittances in parallel; (c) plot a curve of the flux density distribution on the surface of one of the cylinders.

Prob. 2. Show that on an equipotential surface surrounding A, and consequently on the corresponding metal surface, the flux density varies inversely as r_1^2.

Prob. 3. Show how to calculate the permittance between a large cylinder and a given infinite plane.

Prob. 4. Show that A and B are *inverse points* with respect to any equipotential circle; this means that the radius qC is the geometric mean between the distances qB and qA, and the radius pK is the geometric mean between the distances pA and pB. This is true whether the radii qC and pK are equal or not.

Prob. 5. Extend the theory given in this article to the calculation of the elastance and flux density distribution between two large spheres. Consult the chapters on electrostatics in some standard work on the mathematical theory of electricity and magnetism.

CHAPTER XVII

EQUIVALENT ELASTANCE AND CHARGING CURRENT OF THREE-PHASE LINES

64. Three-phase Line with Symmetrical Spacing. Consider an unloaded three-phase line, and let the three conductors be denoted by A, B, and C respectively. There is a displacement of electricity between each pair of conductors, and since the three instantaneous voltages are different, the displacements between the three pairs of conductors at any instant are also different. The three sets of lines of force are relatively displaced and the flux density varies from instant to instant, so that there is produced in reality a revolving electrostatic field. Let the instantaneous displacements which issue from the three conductors per unit of axial length be denoted by q_1, q_2, and q_3, where the subscripts 1, 2, and 3 refer to the conductors A, B, and C respectively. To be consistent with the notation used before, these symbols should be provided with the "prime" sign, but this sign is omitted in order not to obscure the formulæ. The displacements are considered positive when they are directed from the conductors into the dielectric. Since electricity behaves like an incompressible fluid, as much of it as is displaced at any instant out of one conductor must be displaced into the other two conductors, so that at all times the following relation holds, namely,

$$q_1 + q_2 + q_3 = 0. \quad \quad (325)$$

The three q's vary with the time according to the sine law. With a symmetrical spacing of the wires, and symmetrical voltages forming an equilateral triangle (Fig. 53), the effective values of the three q's are equal, and the corresponding instantaneous values are displaced in time phase by 120 degrees. The charging current per unit length of a conductor is equal to the rate of change of the corresponding displacement with the time, or

$$i = dq/dt. \quad \quad (326)$$

But, with sinusoidal voltages, the displacements vary also according to the sine law, or

$$q = Q_m \sin 2\pi ft, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (327)$$

where Q_m is the maximum value of the displacement from one of the conductors. Substituting this value of q in eq. (326), we find that

$$i = 2\pi f Q_m \cos 2\pi ft. \quad . \quad . \quad . \quad . \quad . \quad . \quad (328)$$

Consequently, the amplitude of the charging current

$$I_m = 2\pi f Q_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (329)$$

and the same relation holds true for the *effective* values of the displacement and current. It is to be noted that the charging current leads the flux by 90 electrical degrees. Thus, knowing the displacement, the charging current can be calculated from eq. (329). If Q_m is expressed in microcoulombs per kilometer, I_m is in microamperes per kilometer.

The actual charging current which flows through a cross-section of the conductor, is equal to that necessary to supply the displacement between this cross-section and the receiver end of the line. In other words, the charging current varies along the line, from a maximum at the generator end to zero at the receiver end. If the effective voltage along the line were constant in phase and magnitude, the amplitude of the charging current would vary according to a straight-line law. In reality, the voltage varies along the line, due to its resistance and inductance, so that the variations in phase and amplitude of the charging current along the line follow a much more complicated law.

The influence of the permittance of the line upon its voltage regulation is treated in Arts. 68 and 69 below. The problem here is a preliminary one; namely, with a given size and arrangement of conductors in a three-phase line, to find the permittance per kilometer of the equivalent single-phase line, for which the voltage regulation is usually calculated. The problem is solved by applying again the principle of superposition. Each conductor is considered as forming a condenser with a concentric cylinder of infinite radius, the three phases being star-connected and the three cylinders grounded. The vectors of the star and delta voltages are shown in Fig. 53, the subscripts 1, 2, 3 referring again to the conductors A, B, and C respectively.

Applying eq. (290) for the voltage between the conductors A and B, we have, for instantaneous values,

$$e_{12} = (\sigma q_1/2\pi)\,\mathrm{Ln}\,(b/a) + (\sigma q_2/2\pi)\,\mathrm{Ln}\,(a/b), \quad . \quad (330)$$

where, as before, the spacing is denoted by b, and the radii of the conductors by a. The first term on the right-hand side of this equation represents the action of system A, the second term that of system B. The action of the system C is equal to zero, because, on applying eq. (290) for this system, it is observed that $r = r'$, on account of the symmetrical spacing. In other words,

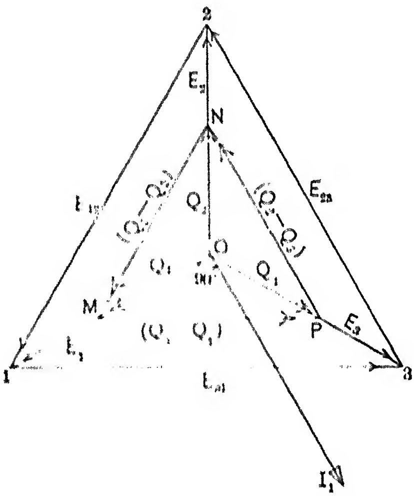

FIG. 53. Electric displacements in a three-phase line with *symmetrical voltages* and *symmetrical spacing*.

for system C the conductors A and B lie on the same equipotential cylindrical surface. The preceding equation is simplified to $e_{12} = (q_1 - q_2)S'$, where S' is the elastance expressed by eq. (296), that is, the elastance between one of the conductors and the plane of symmetry OO', as if the third conductor did not exist. Owing to symmetry, the other two equations are similar; thus we have

$$\left.\begin{aligned} e_{12} &= (q_1 - q_2)S'; \\ e_{23} &= (q_2 - q_3)S'; \\ e_{31} &= (q_3 - q_1)S'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (331)$$

This result is interpreted graphically in Fig. 53, remembering that relations which hold true algebraically for instantaneous values of sinusoidal quantities, hold true geometrically for the corresponding vectors of these quantities. According to eqs. (331), the instantaneous values of $(q_1 - q_2)$, $(q_2 - q_3)$, and $(q_3 - q_1)$ are in phase with the corresponding voltages e_{12}, e_{23}, and e_{31}. For this reason, the vectors $(Q_1 - Q_2)$, $(Q_2 - Q_3)$, and $(Q_3 - Q_1)$ are drawn in phase with the vectors E_{12}, E_{23}, and E_{31}. In regard to the quantities Q_1, Q_2, and Q_3, we know that, for reasons of symmetry, they are equal numerically and are displaced in phase relatively to each other by 120 degrees. Therefore, they must be represented by vectors from the center O to the vertices of the triangle MNP. The condition is then fulfilled that each side of this triangle is equal to the difference of two vectors from the point O.

We see now that the three electric displacements Q_1, Q_2, and Q_3 are in phase with the corresponding star- or Y-voltages of the system; also, from the similarity of the triangles, we have $E_{12}/E_1 = (Q_1 - Q_2)/Q_1$, with corresponding relations for the other two phases. Consequently, eqs. (331) are reduced simply to

$$\left.\begin{aligned} E_1 &= Q_1S'; \\ E_2 &= Q_2S'; \\ E_3 &= Q_3S'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (332)$$

We thus arrive at the following important conclusion: *The displacement* (and consequently the charging current) *per phase of a three-phase line with symmetrical spacing and symmetrical voltages is equal to that in a single-phase line with the same conductors and the same spacing, provided that the star voltage of the three-phase line is equal to that between one conductor and the neutral plane OO' in the single-phase line.*

As explained in Art. 36, an equivalent single-phase line is obtained by taking one conductor of the three-phase line and assuming the transmission voltage to be equal to the star voltage of the actual transmission line; the return conductor is supposed to be devoid of both resistance and inductance. The preceding rule gives a simple method for finding the permittance of the equivalent line; namely, *the permittance of the equivalent single-phase line is equal to that between one of the conductors of the actual line and the plane of symmetry between it and one of the other conductors, as if the third conductor did not exist.*

The calculation of the charging current with an unsymmetrical spacing of conductors is much more involved, and is explained in the next article. Fortunately, however, the spacing between the conductors affects the value of the charging current but little, with the usual ratios between size of conductor and spacing. The student can easily verify this fact by consulting any available table of capacities or charging currents of transmission lines. The reason for this is that the principal part of the elastance between two small conductors occurs near the conductors, where the flux density is comparatively high. Consequently, it is possible in practice to estimate the permittance per phase of a three-phase line with unsymmetrical spacing, by finding the limits of the permittance with symmetrical spacings. For instance, let two conductors be placed on a cross-arm and the third on top of the pole, forming an isosceles triangle. Let the spacings be 2 m. and 1.6 m. respectively. The charging currents are different in the three conductors, but the average value is larger than with a symmetrical spacing of 2 m., and smaller than with a symmetrical spacing of 1.6 m. Having found the charging currents or the equivalent permittances for these two spacings, one can assume an intermediate value by interpolation, or else take one of the two limits, whichever gives the more unfavorable operating conditions of the line.

It is rather a tedious problem to estimate the influence of the ground upon the charging currents in a three-phase line. The theory is simple, the ground being replaced by the images of the three conductors, as in Fig. 51; but the formulæ are long and involved, because the effects of six separate systems must be superimposed. See problem 3 in the next article.

Prob. 1. Show that when one of the conductors in a three-phase line fails, the charging current in the other two conductors drops to 86.6 per cent of its former value. Solution: Let C be the permittance between one of the conductors and the plane of symmetry between it and one of the other conductors. Then the charging current with the three phases alive is $kC(E/\sqrt{3})$, where E is the line voltage, and k is a coefficient of proportionality with which we are not concerned here. Operating single-phase, the charging current is $kC(\frac{1}{2}E)$. The ratio of the two is $0.5/(1/\sqrt{3}) = 0.866$.

Prob. 2. A three-phase, 140-kv., 25-cycle transmission line consists of conductors 2 cm. in diameter; the spacing is symmetrical and equal to 3.5 m.; the length of the line is 250 km. What is the total reactive

power necessary to keep the line alive, and what are the voltage and the permittance per kilometer of the equivalent single-phase line?

Ans. 7270 kva.; 80.8 kv.; 0.00947 mf. per km.

Prob. 3. A three-phase transmission line consists of conductors 18 mm. in diameter, suspended all three in the same vertical plane, at a distance of 2.4 m. between the adjacent conductors. What are the limits of the elastance of the equivalent single-phase line?

Ans. 100 and 113 megadarafs per km.

Note: The proximity of the two limits shows that it is sufficient for practical purposes to consider the symmetrical spacing only, as far as the dielectric and magnetic effects are concerned. Mr. J. G. Pertsch, Jr., has called the author's attention to the fact that, with certain simplifying assumptions, and when the three wires are transposed, the equivalent spacing for inductance and capacity is equal to the geometric mean of the three actual spacings, or

$$b_{eq} = \sqrt[3]{b_{12}b_{23}b_{31}}.$$

In the case under consideration the equivalent spacing is 3.02 m., and the corresponding elastance equals 105 mgd. per km.

Prob. 4. Extend the treatment given in this article to the case in which the three delta voltages are different (Fig. 54), and show that the point O coincides with the center of gravity of the triangle, a symmetrical spacing of the conductors being presupposed as before. Solution: Equations (331) hold true as before, and the sides of triangle MNP are parallel to those of 123, but the point O cannot be determined in this case from the symmetry of the figure. Any point O within the triangle 123 gives a set of star voltages E_1, E_2, and E_3, which will produce the given set of delta voltages; but there is only one point O from which the rays to the vertices of triangle MNP satisfy condition (325). Since the displacements in the equivalent single-phase lines must be proportional to the voltages, condition (325) requires that the geometric sum of E_1, E_2, and E_3 shall equal zero. The parallelogram $O21'3$ gives the geometric sum of E_2 and E_3 equal to $O1'$. If O is the correct point, $O1'$ must be equal and opposite to $O1$, and $Om = \frac{1}{2}O1$. Similarly, the condition must be fulfilled that $On = \frac{1}{2}O2$, and $Op = \frac{1}{2}O3$. It is known from elementary geometry that the three bisectors of a triangle divide each other in the ratio of 2 to 1, and that the point of their intersection is the center of gravity of the triangle. Hence the point O is the center of gravity of the triangle 123.

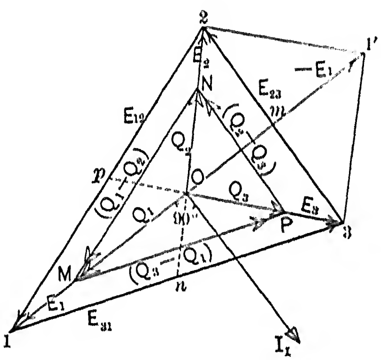

FIG. 54. Electric displacements in a three-phase line with *unsymmetrical voltages* and *symmetrical spacing*.

From eqs. (331) we again derive eqs. (332), and finally arrive at the same conclusion as that printed in italics after these equations. The three star voltages being different, one from another, the three charging currents are also different, each leading the corresponding Q by 90 degrees. The permittance and the voltage of the equivalent single-phase line are also different for each phase, in spite of the symmetrical spacing of the conductors.

Prob. 5. For a given three-phase line with symmetrical spacing and voltages, draw the electrostatic field for the instant when one of the delta voltages is equal to zero; also make three drawings of the field at the ends of intervals $\frac{1}{12}$, $\frac{2}{12}$ and $\frac{3}{12}$ of a cycle later. Use the principle of superposition explained in Prob. 5, Art. 62, and apply it to the three component systems, A, B, and C, keeping in mind the relative magnitudes of the instantaneous displacements.

65. Three-phase Line with Unsymmetrical Spacing.[1] As is mentioned in the preceding article, the calculation of charging currents in a three-phase line with unsymmetrical spacing is much more involved than with symmetrical spacing, and is not of much practical importance at present. An outline of it is given here in order to fix more firmly in the student's mind the general principle of superposition, and the method by which the results are derived in the preceding article. Moreover, the influence of the dielectric is becoming more and more important, as the transmission voltages and the lengths of transmission lines are increased. The time may come when the exposition given in this article will be of assistance in the solution of practical problems.

Let the three spacings be denoted by b_{12}, b_{23}, and b_{31} respectively. Equations (325) to (329) inclusive hold true as with a symmetrical spacing, but eq. (330) now becomes

$$e_{12} = (\sigma q_1/2\pi)\,\mathrm{Ln}\,(b_{12}/a) + (\sigma q_2/2\pi)\,\mathrm{Ln}\,(a/b_{12}) + (\sigma q_3/2\pi)\,\mathrm{Ln}\,(b_{23}/b_{31}), \quad \ldots \quad (333)$$

because the effect of the system C is not equal to zero in this case. Similar equations may be written for e_{23} and e_{31}, but only two equations are independent; the third is obtained by combining the two others, because the third voltage in a delta combination is determined by the other two voltages. The third independent equation is (325), and these three equations determine the three unknown q's.

[1] This article may be omitted, if so desired.

The following solution of these equations gives an insight into the physical relations, and leads to a result which is convenient in numerical work. The last term on the right-hand side of eq. (333) is usually much smaller than the other two terms, so that it may be conveniently represented in the form of a correction to the other two, thus preserving the general form of eq. (330). Substituting the value of q_3 from eq. (325) into (333), we obtain

$$e_{12} = (\sigma q_1/2\pi)\,\mathrm{Ln}\,(b_{c1}/a) - (\sigma q_2/2\pi)\,\mathrm{Ln}\,(b_{c2}/a), \quad . \quad (334)$$

where the quantities,

$$\left.\begin{aligned} b_{c1} &= b_{12}b_{31}/b_{23},\\ b_{c2} &= b_{23}b_{12}/b_{31},\\ b_{c3} &= b_{31}b_{23}/b_{12}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (335)$$

may be called the corrected spacings. The factors by which the displacements q_1 and q_2 are multiplied in eq. (334) are familiar, since they are of the same form as the right-hand member of eq. (296). It will be recalled that eq. (296) expresses the elastance between one conductor and the plane of symmetry of a single-phase line. The above-mentioned factors therefore represent the elastances of single-phase lines having the corrected spacings b_{c1} and b_{c2} respectively. Denoting the reciprocals of these elastances, or the corrected permittances, by C' with the corresponding subscripts, eq. (334) and the two similar equations for the other phases are reduced to the form

$$\left.\begin{aligned} e_{12} &= q_1/C'_1 - q_2/C'_2;\\ e_{23} &= q_2/C'_2 - q_3/C'_3;\\ e_{31} &= q_3/C'_3 - q_1/C'_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (336)$$

On the other hand, for any star point O, no matter where located, we have the following relation between the delta and star voltages:

$$\left.\begin{aligned} e_{12} &= e_1 - e_2;\\ e_{23} &= e_2 - e_3;\\ e_{31} &= e_3 - e_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (337)$$

We again select the neutral point in such a manner that each star voltage is in phase with the corresponding q; that is, in phase quadrature with the corresponding charging current. Then the given three-phase system is directly resolved into three independent single-phase lines, and our problem is solved. If the

point O is so selected, then by comparing eqs. (336) and (337) we have

$$\left.\begin{aligned} e_1 &= q_1/C_1; \\ e_2 &= q_2/C_2; \\ e_3 &= q_3/C_3. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (338)$$

Substituting the values of the q's from these equations into eq. (325) gives

$$C_1e_1 + C_2e_2 + C_3e_3 = 0. \quad \ldots \ldots \quad (339)$$

This is the condition which the point O must satisfy if eqs. (338) are to hold true. Eliminating e_2 and e_3 from eq. (339) by means of the first and the last of the eqs. (337), and solving for e_1, we obtain

$$e_1 = e_{12}C_2/C - e_{31}C_3/C, \quad \ldots \ldots \quad (340)$$

where

$$C = C_1 + C_2 + C_3. \quad \ldots \ldots \quad (341)$$

As mentioned above, C_1, C_2, and C_3 are the permittances per unit length between one wire and the plane of symmetry, for the corrected spacings defined by eqs. (335).

Since relations which hold true algebraically for instantaneous values also hold true geometrically for the vectors of the same quantities, eq. (340) suggests a simple method for finding graphically the position of the neutral point O in the vector diagram (Fig. 55). To locate O, plot $1k = E_{12}C_2/C$ in the direction opposite to E_{12}, and $kO = E_{31}C_3/C$ parallel to E_{31}. Or else, the problem may be solved analytically, using either the orthogonal or the trigonometric form of complex quantities. Having determined the position of O, the three star voltages become known, and then the corresponding displacements are found from eqs. (338). The charging currents are determined by eq. (329), and are in leading quadrature with the corresponding star voltages. The given system is thus resolved into three independent equiva-

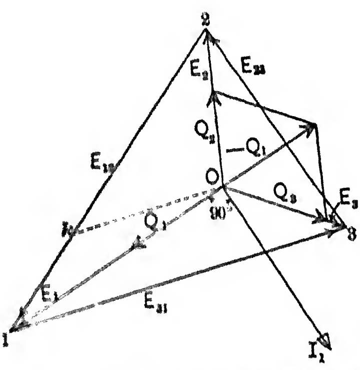

FIG. 55. Electric displacements in a three-phase system with *unsymmetrical voltages* and *unsymmetrical spacing*.

lent single-phase systems with the voltages E_1, E_2, and E_3, and the permittances C_1, C_2, and C_3, per unit length.

Instead of using the treatment given above, one could find the equivalent conductance and susceptance by using a method analogous to that employed in Art. 63 of the *Magnetic Circuit*.

Prob. 1. Determine the actual equivalent elastances in problem 3 of the preceding article, and compare them with the assumed limits.

Prob. 2. Extend the treatment given above to the case in which the cross-sections of the three conductors are different, one from another.

Prob. 3. Show how to estimate the influence of the ground upon the charging currents in a three-phase line, using the method of successive approximations. Solution: Replace the conducting ground by the three images A', B', and C' of the actual conductors, as in Fig. 51. This gives six electric systems with cylinders at infinity. Applying the principle of superposition to the voltages between the conductors $A-B$ and $B-C$, we get, by analogy with eq. (333):

$$\left.\begin{aligned} e_{12} = {} & (\sigma q_1/2\pi)\,\mathrm{Ln}\,(b_{12}/a) + (\sigma q_2/2\pi)\,\mathrm{Ln}\,(a/b_{12}) + (\sigma q_3/2\pi)\,\mathrm{Ln}\,(b_{23}/b_{31}) \\ & - (\sigma q_1/2\pi)\,\mathrm{Ln}\,(BA'/AA') - (\sigma q_2/2\pi)\,\mathrm{Ln}\,(BB'/AB') \\ & - (\sigma q_3/2\pi)\,\mathrm{Ln}\,(BC''/AC''); \\ e_{23} = {} & (\sigma q_2/2\pi)\,\mathrm{Ln}\,(b_{23}/a) + (\sigma q_3/2\pi)\,\mathrm{Ln}\,(a/b_{23}) + (\sigma q_1/2\pi)\,\mathrm{Ln}\,(b_{31}/b_{12}) \\ & - (\sigma q_2/2\pi)\,\mathrm{Ln}\,(CB'/BB') - (\sigma q_3/2\pi)\,\mathrm{Ln}\,(CC''/BC'') \\ & - (\sigma q_1/2\pi)\,\mathrm{Ln}\,(CA'/BA'). \end{aligned}\right\} \qquad (342)$$

From these two equations, together with eq. (325), the three unknown q's can be evaluated. By a method similar to that used in the text above, eqs. (342) are conveniently reduced to the form

$$\left.\begin{aligned} & e_{12} + (\sigma/2\pi)\,[q_1\,\mathrm{Ln}\,(BA'/AA') + q_2\,\mathrm{Ln}\,(BB'/AB') + q_3\,\mathrm{Ln}\,(BC''/AC'')] \\ & \qquad = q_1/C_1 - q_2/C_2; \\ & e_{23} + (\sigma/2\pi)\,[q_2\,\mathrm{Ln}\,(CB'/BB') + q_3\,\mathrm{Ln}\,(CC''/BC'') + q_1\,\mathrm{Ln}\,(CA'/BA')] \\ & \qquad = q_2/C_2 - q_3/C_3; \end{aligned}\right\} \qquad (343)$$

where the three C's and the corrected spacings are the same as before, without the ground. The last three terms on the left-hand side of eqs. (343) are small as compared to e_{12} and e_{23}, and represent the effect of the ground. Therefore, the simplest way of solving these equations is to neglect the correction terms in the first approximation, and to solve for the three q's exactly as explained in the text above, by finding the proper point O (Fig. 55). The correction terms may be said to modify the values of e_{12} and e_{23} in eqs. (343). Having the values of the q's in the first approximation, the corrections are calculated, and, being added to e_{12} and e_{23}, give new values of the latter, say e'_{12} and e'_{23}. Having thus modified the triangle 123 in Fig. 55, a new point O is found, and new values of the q's. The corrections for e_{12} and e_{23} can now be determined more accurately, and then new values of the q's found, which will be more nearly correct than the foregoing ones. In this way, the influence of the ground can be estimated with any desired degree of accuracy, without solving long and involved simultaneous equations.

CHAPTER XVIII

DIELECTRIC REACTANCE AND SUSCEPTANCE IN ALTERNATING-CURRENT CIRCUITS

66. Dielectric Reactance and Susceptance. Let a condenser of permittance C, or elastance S, be connected across an alternating-current line of voltage E and frequency f. Let any instantaneous value of the voltage be denoted by e, where $e = E_m \sin 2\pi ft$; then the corresponding instantaneous displacement in the dielectric is

$$q = eC = e/S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (344)$$

This displacement varies according to the sine law and is in phase with the voltage, because q is at every instant proportional to e. The charging current flowing from the line into the condenser is at any instant equal to the rate of change of q with the time, or

$$i = dq/dt = 2\pi fCE_m \cos 2\pi ft. \quad . \quad . \quad . \quad . \quad (345)$$

It will be seen from this equation that the charging current leads the voltage by 90 degrees, as has already been explained in Art. 48 above. The amplitude of the charging current is

$$I_m = 2\pi fCE_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (346)$$

It may thus be said that a permittance C connected across a source of voltage, of frequency f, is equivalent to a susceptance

$$b = -I_m/E_m = -2\pi fC. \quad . \quad . \quad . \quad . \quad . \quad . \quad (347)$$

The minus sign is necessary because the current is leading, while with a magnetic susceptance it is lagging. In other words, by using the minus sign in the case of *dielectric susceptance*, and the plus sign for magnetic susceptance, it is possible to extend the formulæ deduced in Chapters 8 and 9 to alternating-current circuits containing dielectrics.

In the preceding formulæ C is in farads, S in darafs, q in coulombs, and b in mhos. If C is expressed in microfarads and S in megadarafs, eq. (347) becomes

$$b = -2\pi fC \times 10^{-6} = -2\pi f \times 10^{-6}/S \text{ mhos.} \quad . \quad . \quad (348)$$

The corresponding dielectric reactance is

$$x = -10^6/(2\pi fC) = -10^6 S/(2\pi f) \text{ ohms.} \quad . \quad . \quad (349)$$

The dielectric susceptance is equal to $-2\pi fC$ only when there is no resistance in series with the condenser. When a condenser is connected in series or in parallel with an ohmic resistance, the treatment is analogous to that of a magnetic inductance in combination with a resistance; that is, equivalent series and parallel combinations are used, as explained in Art. 27. To give a detailed treatment here would simply be to repeat what has already been explained in the above-mentioned article. The only difference is that expressions (348) and (349) are used in place of (107) and (86), and the currents are leading, while with magnetic reactance they are lagging with respect to the impressed voltage.

In some circuits both magnetic and dielectric susceptances are connected in parallel. They are simply added, taking into consideration their opposite signs. For instance, a magnetic susceptance of 7 mhos in parallel with a dielectric susceptance of 5 mhos is equivalent to a net magnetic susceptance of 2 mhos. A similar rule is applied when magnetic and dielectric reactances are connected in series.

With these explanations, the student will have no difficulty in dealing with any combination of resistances, condensers, and inductance coils in an alternating-current circuit.

Prob. 1. A condenser of 7.3 mf. permittance is connected across a 500-volt, 60-cycle supply. What are the susceptance and the charging current?

Ans. $b = -0.002754$ mho; $I = j\,1.377$ amp., the voltage being the reference vector.

Prob. 2. The condenser in the preceding problem is shunted by a non-inductive resistance of 750 ohms. Find the total current and the power-factor. Solution: The current through the resistance is $= 500/750 = 0.6667$ amp.; $\tan\phi = 1.377/0.6667 = 2.065$; $\cos\phi = 43.58$ per cent (leading). Total current $= 0.6667/0.4358 = 1.53$ amp.

Prob. 3. The condenser and the resistance in the preceding problem are connected in series, instead of in parallel. What is the equivalent parallel combination?

Ans. $C_p = 1.387$ mf.; $r_p = 926$ ohms.

Prob. 4. The voltage at the receiver end of a 25-cycle, single-phase transmission line is $45 + j\,57$ kv.; the load current is $178 + j\,69$ amp. The series magnetic impedance of the line is $32 + j\,68$ ohms, and its capacity is 4.24 mf. Calculate the generator current and voltage. For purposes of calculation, one half of this capacity can be assumed to be

connected across the generator end of the line, the other half across the receiver end. Solution: The dielectric susceptance at the receiver end of the line is $-2\pi \times 25 \times 2.12 \times 10^{-6} = -0.333 \times 10^{-3}$ mho. The corresponding charging current is

$$j\,0.333 \times 10^{-3}(45000 + j\,57000) = -19 + j\,15 \text{ amp.}$$

Consequently the total line current is $159 + j\,84$ amp. The line drop is $(159 + j\,84)(32 + j\,68) = -624 + j\,13500$ volts. The generator voltage is $44.38 + j\,70.5$ kv. The charging current at the generator end is $j\,0.333\,(44.38 + j\,70.5) = -23.5 + j\,14.79$ amp. The generator current is $135.5 + j\,98.8$ amp.

Prob. 5. Explain the physical reason why a dielectric susceptance increases with the frequency, while a magnetic susceptance is inversely proportional to it.

Prob. 6. Investigate the influence of a condenser in a circuit to which a non-sinusoidal voltage is applied; give a treatment similar to that in Art. 23. Show that the presence of an elastance accentuates higher harmonics in the current, while an inductance tends to diminish them. Make the physical reason for this difference clear to yourself.

67. Current and Voltage Resonance. Let a condenser be connected *in parallel* with a pure reactance coil, across an alternating-current line. Let the current through the condenser be 5 amp., leading, and that through the coil, 3 amp., lagging. Then the total current supplied from the generator is 2 amp., leading. Thus, we have the paradox that the resultant current is smaller than either of its components. It is even possible to adjust the permittance and inductance to such values as to make the leading and lagging components equal, in which case the generator current is zero. This condition is called *current resonance.* When the line current is reduced to zero, total or perfect resonance takes place; otherwise the resonance is called partial. The condition for perfect resonance is that the lagging current shall be equal to the leading current, or, what is the same, the dielectric susceptance must be numerically equal to the magnetic susceptance. Thus, if there is no resistance in either circuit,

$$2\pi fC = 1/(2\pi fL),$$

from which

$$2\pi f\sqrt{CL} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (350)$$

From this equation, any one of the three quantities f, C, and L can be determined, when the other two are given. Condition (350) may be fulfilled for the frequency of one of the higher har-

monics of an e.m.f. wave, in which case we have partial resonance for the fundamental wave, and perfect resonance for one of the harmonics. If such is the case, the line current does not contain this harmonic, although it may be present to a considerable amount in the two branch currents.

From the point of view of energy, current resonance consists in a periodic transformation of the potential energy of the electrostatic field into the kinetic energy of the magnetic field, and vice versa. When the current is at its maximum, the energy of the magnetic field of the reactance coil is also a maximum. But at this moment the voltage, and consequently the electrostatic displacement, are equal to zero, so that the whole energy of the circuit is in the magnetic field. One quarter of a cycle later, the displacement and the stored energy in the condenser are at a maximum, but the current and the magnetic field are equal to zero. At intermediate moments, the energy is contained partly in the electrostatic, and partly in the magnetic field. When condition (350) is satisfied, the maxima of the two energies are numerically equal, and the system "oscillates" freely in the electrical sense, in a manner analogous to the swinging of a pendulum. The generator merely maintains the necessary frequency, and supplies the i^2r loss. Without this loss, it would not be necessary to have the generator at all; the oscillations, once started, would continue indefinitely at the proper frequency. When the two energies are not equal, there must be a cyclic exchange of energy between the generator and one of the branches; namely, the one whose storage capacity for energy, at the generator frequency, is larger than that of the other branch. We then have partial current resonance.

The presence of resistance in either branch obscures the effect of resonance to some extent, leaving, however, its general character unchanged. The best way to see the influence of resistance is to replace each impedance by its equivalent parallel combination. We then have two pure susceptances with reactive currents, and two conductances, the currents through which are in phase with the line voltage. The energy supplied to the conductances is converted into heat, and thus does not enter into the electrical oscillations.

Let now a dielectric reactance be connected *in series* with a magnetic reactance, across an alternating-current line. The cur-

rent through the two devices is the same, and may be taken as the reference vector. Let the dielectric reactance be such as to produce across the condenser a drop of 1000 volts, lagging behind the current by 90 electrical degrees. Let the voltage across the reactance coil be equal to 900 volts, leading the current by 90 degrees. With these conditions, the total line voltage is equal to 100 volts, lagging behind the current by 90 degrees. Thus, with a line voltage of only 100, it is possible to produce partial voltages of 1000 and 900 respectively. This condition is called *voltage resonance*. When the two reactances in series are equal, we have complete voltage resonance; otherwise the resonance is partial. The student will readily see that the condition for complete voltage resonance is also expressed by eq. (350). In this case, the presence of resistance has no effect upon the correctness of the equation. By reading again the foregoing discussion of current resonance, and applying it to voltage resonance, the points of similarity and the differences between the two will be easily seen.

One has to be on guard against possible resonance and a dangerous rise in potential in the operation of transmission lines and extended cable systems, because there the presence of permittance and inductance offers favorable conditions for surges between the dielectric and magnetic energies. These surges either produce large currents which open the circuit-protecting devices and interrupt the service, or the potential is raised to a value at which the insulation of the system is broken down. With a clear understanding of the principle of interchange of energy explained above, the student ought to be able to follow without difficulty special works on the subject.[1]

Prob. 1. A magnetic reactance of 65 ohms is connected in parallel with a permittance of 73.6 mf., across a 2200-volt, 25-cycle circuit. Determine the total current, and the component currents, through the reactance and through the condenser.

Ans. 33.85; 25.44; 8.41 amp. (lagging). This is a case of partial current resonance, the total current being smaller than one of its components.

[1] See W. S. Franklin, *Electric Waves*; C. P. Steinmetz, *Electric Discharges, Waves, and Impulses*, also his larger work on *Transient Electric Phenomena and Oscillations*. Some elementary experiments and curves of resonance will be found in V. Karapetoff's *Experimental Electrical Engineering*, Vol. II, Arts. 440 to 445.

Prob. 2. The permittor and the reactance coil given in the preceding problem are connected across the same line in series, instead of in parallel. Find the total current and the component voltages.

Ans. 102.3 amp. (leading); 2200 = 8850 − 6650 volts. This is a case of partial voltage resonance, the voltage drop across each of the two devices being larger than the applied voltage.

Prob. 3. The elastance of a 60-cycle underground cable system is equal to 11.5 kilodarafs; at what value of the inductance in the circuit is resonance of the seventh harmonic to be feared?

Ans. 1.65 millihenry.

68. Voltage Regulation of a Transmission Line, Taking Its Distributed Permittance into Account. The voltage regulation of a transmission line, disregarding its permittance, is treated in Art. 33. The value of the permittance of a single-phase line is deduced in Art. 60, while in Chapter 17 the effect of the charging current in a three-phase line is considered, and it is shown how to calculate the permittance of the equivalent single-phase line. The inductance of transmission lines is treated in Chapter 11 of the author's *Magnetic Circuit.* It remains now to show how to determine, for a given load, the relation between the generator and receiver voltages of an equivalent single-phase line, knowing its constants; viz., the values of the distributed resistance, magnetic reactance, and dielectric susceptance.

Let the total resistance of the equivalent single-phase line be r ohms, and its magnetic reactance x ohms. Then the series impedance of the line is

$$Z = r + jx. \qquad (351)$$

Let the dielectric susceptance of the line be b mhos, where b, according to eq. (347), is a negative quantity; and let the leakage conductance to the ground be g mhos. Then the shunted admittance of the line is

$$Y = g - jb. \qquad (352)$$

The leakage conductance is due to imperfect insulation of the line, and may also be made to take into account the corona loss, if any exists. The value of g can only be estimated, and in most cases may be safely neglected. It is introduced here in order to obtain a more general result, at the same time making the expressions for Z and Y symmetrical.

Since Y is uniformly distributed along the line, the current changes as the distance from the generator increases; and there-

fore it is necessary to consider the electrical relations in an infinitesimal length ds, at some intermediate point of the line. Let the line voltage at this point be $\underset{.}{E}$, and the line current, $\underset{.}{I}$. The series impedance of the element ds is $(Z/l)\,ds$, and its shunted admittance is $(Y/l)\,ds$, where l is the total length of the line. Let $d\underset{.}{E}$ be the increment in the voltage in the length ds, and let $d\underset{.}{I}$ be the corresponding increment in the line current due to the shunted admittance. We have then

$$d\underset{.}{E} = -\underset{.}{I}(Z/l)\,ds, \quad . \quad . \quad . \quad . \quad . \quad (353)$$

and

$$d\underset{.}{I} = -\underset{.}{E}(Y/l)\,ds. \quad . \quad . \quad . \quad . \quad . \quad (354)$$

The minus sign is needed on the right-hand side of eq. (353), because the drop in voltage $\underset{.}{I}(Z/l)\,ds$ causes a decrement in $\underset{.}{E}$. Likewise the charging current $\underset{.}{E}(Y/l)ds$ causes a decrement in the line current.

Equations (353) and (354) contain two dependent variables, $\underset{.}{E}$ and $\underset{.}{I}$. To eliminate $\underset{.}{I}$, we divide both sides of eq. (353) by ds and take the derivative with respect to s. The result is

$$d^2\underset{.}{E}/ds^2 = -(d\underset{.}{I}/ds)\,(Z/l).$$

Substituting the value of $d\underset{.}{I}/ds$ from eq. (354), we obtain

$$d^2\underset{.}{E}/ds^2 = \underset{.}{E}ZY/l^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (355)$$

This is a differential equation of the second order for $\underset{.}{E}$. We shall omit the solution of it and give only the result, for two reasons: first, because most students are not familiar with the methods of integration of differential equations; and secondly, because the solution is most conveniently expressed in hyperbolic functions of a complex variable, a form of function unknown to most students of engineering.[1] Fortunately, even for the longest transmission

[1] The simple theory of hyperbolic functions and the solution of eq. (355) may be found, among others, in the following works and articles: McMahon, *Hyperbolic Functions;* Dr. Kennelly, *Applications of Hyperbolic Functions to Electrical Engineering;* Dr. Steinmetz, *Transient Electric Phenomena;* Pender and Thomson, "The Mechanical and Electrical Characteristics of Transmission Lines," *Trans. Amer. Inst. Electr. Engrs.*, Vol. 30 (1911); W. E. Miller, "Hyperbolic Functions and Their Application to Transmission Line Problems," *General Electric Review*, Vol. 13 (1910), p. 177; M. W. Franklin, "Transmission Line Calculations," *ibid.*, p. 74. For a proof of expansion (356), see Blondel and Le Roy, "Calcul des Lignes de Transport d'Energie à Courants Alternatifs en tenant compte de la Capacité et de la Perditance Reparties," *La Lumière*

lines built or projected, the solution can be represented with sufficient accuracy by a few terms of an infinite series, as follows:

$$\dot{E}_1 = \dot{E}_2(1 + \tfrac{1}{2} YZ + \tfrac{1}{24} Y^2Z^2 + \text{etc.}) + \dot{I}_2 Z(1 + \tfrac{1}{6} YZ + \tfrac{1}{120} Y^2Z^2 + \text{etc.}). \quad (356)$$

In this equation $\dot{E}_1$ is the generator voltage, $\dot{E}_2$ the receiver voltage, the same as in Art. 33, and $\dot{I}_2$ is the load current. Both Y and Z are known complex quantities, and therefore their product and the square of the product are also known. The terms involving Y^2Z^2 are negligibly small in many cases.

Instead of eliminating $\dot{I}$ from eqs. (353) and (354), $\dot{E}$ may be eliminated by a similar process, giving a differential equation for $\dot{I}$, analogous to eq. (355). The solution of this equation is

$$\dot{I}_1 = \dot{I}_2(1 + \tfrac{1}{2} YZ + \tfrac{1}{24} Y^2Z^2 + \text{etc.}) + \dot{E}_2 Y(1 + \tfrac{1}{6} YZ + \tfrac{1}{120} Y^2Z^2 + \text{etc.}), \quad (357)$$

where $\dot{I}_1$ is the generator current, and $\dot{I}_2$ the load current.

The general form of eqs. (356) and (357) is the same as that of the corresponding equations in Art. 33, and in Chapters 11, 12, and 13, so that the methods of calculation indicated there are applicable here, with self-evident modifications.

Neglecting the leakage conductance g in eq. (352), and using the value of permittance given in Art. 60 above, also the value of inductance from Art. 61 of the *Magnetic Circuit*, we find that

$$YZ = (l/1000)^2 \, (-v + jw), \quad . \quad . \quad . \quad . \quad (358)$$

and consequently

$$Y^2Z^2 = (l/1000)^4 \, (v^2 - w^2 - 2\,jvw), \quad . \quad . \quad . \quad (359)$$

where

$$v = 0.09514 \, (0.1\,f)^2 \, [0.46 + 0.05/\log (b/a)], \quad . \quad (360)$$

and

$$w = 0.1515\, fr'/\log (b/a). \quad . \quad . \quad . \quad . \quad . \quad (361)$$

In these expressions, l is the length of the line in kilometers, and r' is the resistance per kilometer of one conductor, in ohms. With

Electrique, Vol. 7, 2nd Series (1909), p. 355; also J. F. H. Douglas, "Transmission Line Calculations," *Electrical World*, Vol. 55 (1910), p. 1066; and Dr. Steinmetz, *Engineering Mathematics*, p. 204. The best tables of hyperbolic functions are those published by the Smithsonian Institution; briefer tables will be found in McMahon's book and in the *General Electric Review*, Vol. 13, supplement to No. 5. See also Seaver's *Mathematical Handbook*, pp. 85 and 266.

the extreme values of the ratio b/a of say 10 and 1000, the value of log (b/a) varies within the narrow limits of 1 to 3, so that the second term in the brackets in eq. (360) is comparatively small. In practice, the value of the whole expression in the brackets in formula (360) is usually between 0.48 and 0.50. This fact is taken advantage of in numerical calculations which do not require particular accuracy.

Prob. 1. Check the numerical coefficients in formulæ (360) and (361).

Prob. 2. For a given receiver voltage, calculate the generator voltage, at no load and at full load, for some very long transmission line, the dimensions of which are taken from a descriptive article.

Prob. 3. Solve problem 2 by the use of tables of hyperbolic functions, following the method indicated in one of the references in the footnote. Compare the results with those obtained in problem 2, and make clear to yourself the relative simplicity, and the limits of accuracy, of the series when one, two, or three terms are used.

69. Approximate Formulæ for the Voltage Regulation of a Transmission Line, Considering Its Permittance Concentrated at One or More Points. Instead of treating the permittance of a transmission line in the correct manner described in the preceding article, it is sometimes assumed to be concentrated at one or more points along the line. The calculation of voltage regulation then becomes similar to the treatment in Chapters 9 to 13. There is no particular advantage in this approximate treatment as far as the simplicity of numerical computations is concerned, because the formulæ obtained are similar to eq. (356). It is advisable, however, for the student to deduce such formulæ in order to see for himself that the form of eq. (356) is a rational one; moreover, this gives him one more exercise in the use of complex quantities.

(a) The simplest assumption is to consider one half of the line permittance (and leakage, if any) concentrated at the generator end of the line, the other half at the receiver end. The load current is in this case apparently increased by the current $\underline{E}_2 \cdot \frac{1}{2} Y$ through the permittance $\frac{1}{2} Y$ connected in parallel with the load, so that the total receiver current is equal to $\underline{I}_2 + \frac{1}{2} \underline{E}_2 Y$. Hence, the generator voltage is

$$\underline{E}_1 = Z(\underline{I}_2 + \tfrac{1}{2} \underline{E}_2 Y) + \underline{E}_2,$$

or

$$\underline{E}_1 = \underline{E}_2(1 + \tfrac{1}{2} YZ) + \underline{I}_2 Z. \quad . \quad . \quad . \quad . \quad (362)$$

Comparing this formula with eq. (356), we see that the principal terms are identical, the difference being in the additional terms containing higher powers of YZ. If the influence of the line permittance is small, for instance in short lines, the results calculated by means of both formulæ differ from each other but very little. There is no reason, however, why the accurate expansion (356) should not be used in all cases, taking as many terms as are required in a given problem.

The generator current, with the capacity concentrated at both ends, is

$$\dot{I}_1 = \dot{I}_2 + \tfrac{1}{2} Y\dot{E}_2 + \tfrac{1}{2} Y\dot{E}_1,$$

or, substituting the value of $\dot{E}_1$ from eq. (362),

$$\dot{I}_1 = \dot{I}_2(1 + \tfrac{1}{2} YZ) + \dot{E}_2 Y(1 + \tfrac{1}{4} YZ). \quad . \quad . \quad (363)$$

This formula is similar to eq. (357), and differs from it only in the values of the coefficients of the minor terms.

(b) The line permittance and leakage may also be concentrated at the middle point of the line, in which case a diagram of connections is obtained similar to Fig. 42, except that the susceptance is dielectric and not magnetic. Introducing the voltage at the center of the line as an auxiliary unknown quantity, and eliminating it from the result, we obtain

$$\dot{E}_1 = \dot{E}_2(1 + \tfrac{1}{2} YZ) + \dot{I}_2 Z(1 + \tfrac{1}{4} YZ), \quad . \quad . \quad (364)$$

and

$$\dot{I}_1 = \dot{I}_2(1 + \tfrac{1}{2} YZ) + \dot{E}_2 Y. \quad . \quad . \quad . \quad . \quad (365)$$

(c) A closer approximation is obtained by assuming a part of the line permittance concentrated at the middle point, and the rest at both ends of the line. The fractions of the total permittance to be assigned to these three points are determined from Simpson's Rule for approximate integration; namely, according to this "parabolic" rule,

$$\begin{aligned} y_{ave} = [1/(3\,n)]\,[&y_0 + 4\,(y_1 + y_3 + \text{etc.} + y_{n-1}) \\ &+ 2\,(y_2 + y_4 + \text{etc.} + y_{n-2}) + y_n], \quad . \quad . \quad . \quad . \quad . \quad (366) \end{aligned}$$

where y_{ave} is the average ordinate of a given curve, n is the number of equal parts into which the total width of the curve is subdivided, and y_0, y_1, etc., are the actual ordinates at the points of division. In the above formula, n must be an even number. Let the given curve represent some arbitrary distribution of the

permittance along the line, and let $n = 2$. The foregoing formula gives

$$C'_{ave} = \tfrac{1}{6}\,(C_0' + 4\,C_1' + C_2'), \quad . \quad . \quad . \quad . \quad (367)$$

where the C's are marked with the prime sign to indicate that they refer to unit length of the line. But in reality the permittance is uniformly distributed over the length of the line, so that $C'_{ave} = C_0' = C_1' = C_2'$. Multiplying both sides of eq. (367) by the length l of the line, we obtain

$$C = \tfrac{1}{6}\,C + \tfrac{2}{3}\,C + \tfrac{1}{6}\,C. \quad . \quad . \quad . \quad . \quad . \quad . \quad (368)$$

This means that two thirds of the total permittance must be concentrated at the middle of the line, and one sixth at each end.[1]

With this distribution of permittance it is again convenient to introduce the voltage at the center of the line as an auxiliary quantity. The relation between the load voltage and the generator voltage is calculated in the well-known manner, by adding the voltage drop in the line to the load voltage. The result is

$$E_1 = E_2(1 + \tfrac{1}{2}\,YZ + \tfrac{1}{36}\,Y^2Z^2) + I_2Z(1 + \tfrac{1}{6}\,YZ); \quad . \quad (369)$$

$$I_1 = I_2(1 + \tfrac{1}{2}\,YZ + \tfrac{1}{36}\,Y^2Z^2) + E_2Y(1 + \tfrac{5}{36}\,YZ + \tfrac{1}{216}\,Y^2Z^2). \quad . \quad . \quad . \quad (370)$$

These formulæ come closer to eqs. (356) and (357) than those obtained in the preceding two approximations.

Prob. 1. Check formulæ (364) and (365) by actually performing the algebraic transformations.

Prob. 2. Check formulæ (369) and (370) by actually performing the algebraic transformations.

Prob. 3. If it be desired to have the permittance concentrated at five equidistant points along the line, show that according to Simpson's Rule one sixth of the total permittance must be placed in the middle, one twelfth at each end, and the rest at one quarter and three quarters of the length of the line.

[1] This result has been first indicated by Dr. Steinmetz, in his *Alternating-Current Phenomena*, in the chapter on "Distributed Capacity."

APPENDIX.

THE AMPERE-OHM SYSTEM OF UNITS.

THE ampere and the ohm can be now considered as two arbitrary fundamental units established by an international agreement. Their values can be reproduced to a fraction of a per cent according to detailed specifications adopted by practically all civilized nations. These two units, together with the centimeter and the second, permit the determination of the values of all other electric and magnetic quantities. The units of mass and of temperature do not enter explicitly into the formulæ, but are contained in the legal definition of the ampere and the ohm. The dimension of resistance can be expressed through those of power and current, according to the equation $P = I^2R$, but it is more convenient to consider the dimension of R as fundamental, in order to avoid the explicit use of the dimension of mass [M]. Besides, there is no direct proof that the physical dimensions of electric power are the same as those of mechanical power. All we know is that the two kinds of power are equivalent one to the other.

For the engineer there is no need of using the electrostatic or the electromagnetic units; for him there is but one *ampere-ohm system*, which is neither electrostatic nor electromagnetic. The ampere has not only a magnitude, but a *physical dimension* as well, a dimension which, with our present knowledge, is fundamental, that is, it cannot be reduced to a combination of the dimensions of length, time, and mass (or energy). Let the dimension of current be denoted by [I] and that of resistance by [R]; let the dimensions of length and time be denoted respectively by the commonly recognized symbols [L] and [T]. The magnitudes and dimensions of all other electric units can be expressed through these four, as shown in the following table. For the corresponding expressions of the magnetic units in the ampere-ohm system, see Appendix I to the author's *Magnetic Circuit.*

TABLE OF ELECTRIC UNITS, AND THEIR DIMENSIONS IN THE AMPERE-OHM SYSTEM

Symbol and Formula	Quantity	Dimensions	Name of the Unit
I	Current	$[I]$	Ampere
$U=I/A$	Current density	$[IL^{-2}]$	Ampere per square centimeter
$Q=IT$	Quantity of electricity and dielectric flux	$[IT]$*	Coulomb (ampere-second)
$D=Q/A$	Dielectric flux density	$[ITL^{-2}]$	Coulomb per square centimeter
$E=IR$	Voltage, difference of potential, or e.m.f.	$[IR]$	Volt
$G=E/l$	Voltage gradient, electric intensity, or dielectric stress	$[IRL^{-1}]$	Volt per centimeter
r	Resistance	$[R]$	Ohm
$x=2\pi fL$	Reactance		
$z=(r^2+x^2)^{\frac{1}{2}}$	Impedance		
$g=1/r_p$	Conductance	$[R^{-1}]$	Mho
$b=1/x_p$	Susceptance		
$y=1/z=(g^2+b^2)^{\frac{1}{2}}$	Admittance		
$\rho=G/U$	Resistivity	$[RL]$	Ohm per centimeter cube
$\gamma=U/G=1/\rho$	Conductivity	$[R^{-1}L^{-1}]$	Mho per centimeter cube
$S=E/Q$	Elastance	$[RT^{-1}]$	Daraf
$C=Q/E=1/S$	Permittance (capacity)	$[R^{-1}T]$	Farad
$\sigma=G/D$	Elastivity	$[RT^{-1}L]$	Daraf per centimeter cube
$\kappa=D/G=1/\sigma$	Permittivity	$[R^{-1}TL^{-1}]$	Farad per centimeter cube
$P=EI$	Power	$[I^2R]$	Watt
$P'=P/V$	Density of power	$[I^2RL^{-3}]$	Watt per cubic centimeter
$W=\frac{1}{2}EQ$	Stored electric energy	$[I^2RT]$	Joule (watt-second)
$W'=\frac{1}{2}GD$	Density of electric energy	$[I^2RTL^{-3}]$	Joule per cubic centimeter
$F=W/l$	Force	$[I^2RTL^{-1}]$	Jouleeen
$L=2W/I^2$	Inductance	$[RT]$	Henry

* These are also the dimensions of the electric pole strength. The concept of pole strength is of no use in electrical engineering, and, in the author's opinion, its usefulness in physics is more than doubtful. The whole elementary theory of electrostatics can and ought to be built up on the idea of stresses and displacements in the dielectric, as is done in this work.

Other units of more convenient magnitude are easily created by multiplying the tabulated units by powers of 10, or by adding the prefixes milli-, micro-, kilo-, mega-, etc.

A study of the physical dimensions of the electric and magnetic quantities is interesting in itself, and gives a better insight into the nature of these quantities. Moreover, formulæ can be checked and errors detected by comparing physical dimensions on both sides of the equation. Let, for instance, a formula for energy be given,

$$W = aQDl/\kappa,$$

where a is a numerical coefficient. Substituting the physical dimensions of all the quantities on the right-hand side of the equation from the table below, the result will be found to be of the dimensions of energy. This fact adds to one's assurance that the given formula is theoretically correct.

A slight irregularity in the system as outlined above is caused by the use of the kilogram as the unit force, because it leads to two units for energy and torque, viz., the kilogram-meter and the joule; 1 kg.-meter = 9.806 joules. *Force ought to be measured in joules per centimeter length*, to avoid the odd multiplier. Such a unit is equal to about 10.2 kg., and could be properly called the *jouleceen* (= 10^7 dynes). There is not much prospect in sight of introducing this unit of force into practice, because the kilogram is too well established in common use. The next best thing to do is to derive formulæ and perform calculations, whenever convenient, in jouleceens, and to convert the results into kilograms by multiplying them by $g = 9.806$.

Thus, leaving aside all historical precedents and justifications, the whole system of electric and magnetic units is reduced to this simple scheme: In addition to the centimeter, the gram, the second, and the degree Centigrade, two other fundamental units are recognized, the ohm and the ampere. All other electric and magnetic units have dimensions and values which are connected with those of the fundamental six in a simple and almost self-evident manner, as shown in the table above.

To appreciate fully the advantages of the practical ampere-ohm system over the C.G.S. electrostatic and electromagnetic systems, one has only to compare the dimensions, for instance, of

current density and voltage gradient in these three systems, as shown below.

	The Ampere-ohm System	C.G.S. Electromagnetic System	C.G.S. Electrostatic System
Dimension of current density...	IL^{-2}	$L^{-\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\kappa^{\frac{1}{2}}$
Dimension of voltage gradient..	IRL^{-1}	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\kappa^{-\frac{1}{2}}$

BIBLIOGRAPHY.

ALTERNATING-CURRENT PHENOMENA, by Chas. P. Steinmetz.

THEORETICAL ELEMENTS OF ELECTRICAL ENGINEERING, by Chas. P. Steinmetz.

ALTERNATING CURRENTS, by Bedell and Crehore.

VECTORS AND VECTOR DIAGRAMS, by Cramp and Smith.

REVOLVING VECTORS, by G. W. Patterson.

ELECTRICAL ENGINEERING, by Thomälen.

DIE WISSENSCHAFTLICHEN GRUNDLAGEN DER ELEKTROTECHNIK, by Benischke.

DIE WECHSELSTROMTECHNIK, by E. Arnold.

PROBLEMS IN ELECTRICAL ENGINEERING, by Waldo V. Lyon.

ELECTRICAL PROBLEMS, by Hooper and Wells.

THE ELEMENTS OF ELECTRICAL ENGINEERING, by Franklin and Esty.

PRINCIPLES OF ELECTRICAL ENGINEERING, by H. Pender.

MODERN VIEWS OF ELECTRICITY, by Oliver Lodge.

ELECTRIC WAVES, by W. S. Franklin.

ELEMENTS OF ELECTROMAGNETIC THEORY, by S. J. Barnett.

KAPAZITAT UND INDUKTIVITAT, by Ernst Orlich.

INDEX